J.R. INSTITUTE OF MATHEMATICS

189/35 BEHIND RAILWAY STATION, VAISH COLLEGE ROAD, ROHTAK PIN-124001 (HARYANA)
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Chapter - 1

Basic Concepts

1.1 Complex numbers

Def. Complex Numbers : A number of the form $x+iy$ is called a complex number and is denoted by $z=x+iy$; where *x* and *y* are real numbers and $i=\sqrt{-1}$.

Def. Conjugate of a complex number : If $z=x+iy$ be a complex number then the conjugate of *z* is denoted by $\bar{z}$, and is defined as $\bar{z}=x-iy$.

Def. Modulus of a complex number : The modulus of a complex number $z=x+iy$ is denoted by $|z|=|x+iy|$ and is defined as the non negative real number given by $|z|=\sqrt{x^2+y^2}$.

Def. Modulus-argument form or polar form or trigonometric form of a complex number : Every non-zero complex number $z=x+iy$ can be put in the form $z=r(\cos\theta+i\sin\theta)$, where r and θ are both real numbers i.e. $x+iy=r(\cos\theta+i\sin\theta)=re^{i\theta}$. The angle θ of inclination of the number *z*, which will always be measured in radians from the positive real axis, is positive when measured counter clockwise and negative when measured clockwise. The angle θ is called an **argument** of *z* and is denoted by $\theta=\arg(z)$.

Def. Principal Argument : The symbol $\arg(z)$ actually represents a set of values, but the argument θ of a complex number that lies in the interval $-\pi<\theta\le\pi$ is called the principal value of $\arg(z)$ or the principal argument of z. The principal argument of *z* is unique and is represented by the symbol $\text{Arg}(z)$ i.e., $-\pi<\text{Arg}(z)\le\pi$. In general, $\arg(z)=\text{Arg}(z)+2n\pi,\ n=0,\pm1,\pm2,.....$

De Moivre's theorem :

(i) If *n* be any integer, positive or negative, then $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$.

(ii) If *n* be a fraction, positive or negative, then one of the values of $(\cos\theta+i\sin\theta)^n$ is $\cos n\theta+i\sin n\theta$.

Properties : Let *z* be any complex number then

1. $\overline{(\bar{z})}=z$

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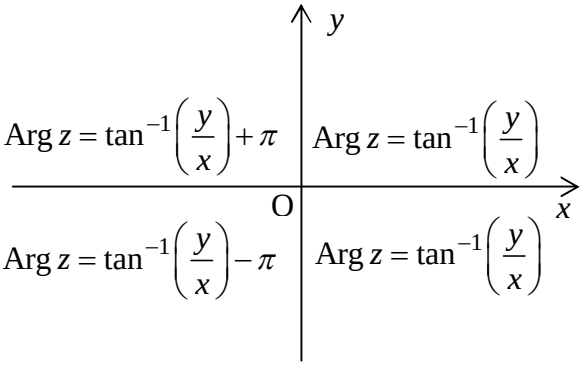
2. If $z = x + iy$, then $z + \bar{z} = 2\,\mathrm{Re}(z)$ and $z - \bar{z} = 2i\,\mathrm{Im}(z)$

3. z is purely real iff $z - \bar{z} = 0$

4. z is zero or purely imaginary iff $z + \bar{z} = 0$

5. $z\bar{z} = |z|^2$

Properties : Let z_1 and z_2 be any complex numbers then

1. $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$
2. $\overline{z_1 - z_2} = \bar{z_1} - \bar{z_2}$
3. $\overline{z_1 z_2} = \bar{z_1}.\bar{z_2}$
4. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}$; provided $z_2 \neq 0$
5. $|z_1 \pm z_2| \leq |z_1| + |z_2|$
6. $|z_1 \pm z_2| \geq \big||z_1| - |z_2|\big|$
7. $|z_1 z_2| = |z_1|.|z_2|$
8. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$.
9. $\arg(\bar{z}) = -\arg(z)$
10. $|z_1 + z_2| = |z_1| + |z_2|$ iff $\arg z_1 - \arg z_2 = 2n\pi$
11. $|z_1 - z_2| = |z_1| + |z_2|$ iff $\arg(z_1) - \arg(z_2) = (2n+1)\pi$
12. $z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$ iff $\arg(z_1) - \arg(z_2) = (2n+1)\frac{\pi}{2}$
13. $z_1\bar{z}_2 - \bar{z}_1 z_2$ iff $\arg(z_1) - \arg(z_2) = n\pi$
14. Parallelogram Law: $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$
15. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$; provided $z_2 \neq 0$
16. In general, $\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$.
17. If $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_2) > 0$, then $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$.
18. The order relations greater than or less than do not apply to complex numbers i.e., the statements $z_1 > z_2$ or $z_1 < z_2$ are meaningless.
19. 

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Exercise 1.1

1. Write the given complex number in polar form and then in the form $a+ib$

(i) $\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)^{12}\left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^{5}$ (ii) $\frac{\left[8\left(\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}\right)\right]^{3}}{\left[2\left(\cos\frac{\pi}{16}+i\sin\frac{\pi}{16}\right)\right]^{10}}$

2. Find the positive integer n for which the equality holds :

(i) $\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{n}=-1$ (ii) $\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{n}=1$

3. Find one value of $\arg z$ when

(i) $z=\frac{-2}{1+\sqrt{3}i}$ (ii) $z=\frac{i}{-2-2i}$ (iii) $z=\left(\sqrt{3}-i\right)^{6}$

4. Write the given complex number in polar form using an argument $\theta\neq\text{Arg}(z)$ and then using $\theta=\text{Arg}(z)$

(i) $-3i$ (ii) $5-5i$ (iii) $\frac{12}{\sqrt{3}+i}$

(iv) $-2-2\sqrt{3}i$ (v) $\frac{3}{-1+i}$ (vi) -10

5. For the complex numbers $z_1=-1$ and $z_2=5i$, verify that :

(a) $\text{Arg}(z_1z_2)\neq\text{Arg}(z_1)+\text{Arg}(z_2)$ (b) $\text{Arg}(z_1/z_2)=\text{Arg}(z_1)-\text{Arg}(z_2)$

Answers

1. (i) $32\left(\cos\frac{13\pi}{6}+i\sin\frac{13\pi}{6}\right);16\left(\sqrt{3}+i\right)$ (ii) $\frac{1}{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right);\frac{i}{2}$

2. (i) $n=6$ (ii) $n=8$

3. (i) $\frac{2\pi}{3}$ (ii) $\frac{5\pi}{4}$ (iii) 11π

4. (i) $3\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right);\ 3\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$

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(ii) $5\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right); 5\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$

(iii) $6\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right);\ 6\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$

(iv) $4\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right); 4\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{-2\pi}{3}\right)\right)$

(v) $\frac{3\sqrt{2}}{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right); \frac{3\sqrt{2}}{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)$

(vi) $10(\cos 3\pi + i\sin 3\pi); 10(\cos\pi + i\sin\pi)$

1.2 Regions and Domains

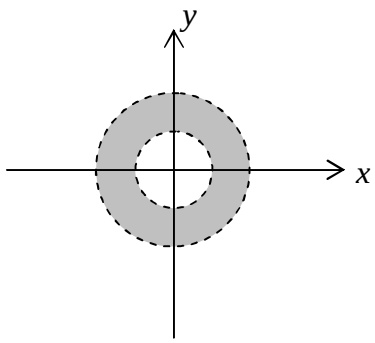
Def. Circle: The equation of a circle with center z_0 and radius r is defined as $|z-z_0|=r$.

Def. Open disc: The equation of an open disc with center z_0 and radius r is defined as $|z-z_0|<r$.

Def. Closed disc: The equation of a closed disc with center z_0 and radius r is defined as $|z-z_0|\le r$.

Def. Punctured disc: The equation of a punctured disc with center z_0 and radius r is defined as $0<|z-z_0|\le r$ or $0<|z-z_0|<r$.

Def. Annulus : The set S_1 of points satisfying the inequality $r_1<|z-z_0|$ lie exterior to the circle of radius r_1 centered at z_0, whereas the set S_2 of points satisfying $|z-z_0|<r_2$ lie interior to the circle of radius r_2 centered at z_0. Thus, if $0<r_1<r_2$, the set of points satisfying the simultaneous inequality $r_1<|z-z_0|<r_2$, is the intersection of the sets S_1 and S_2. This intersection is an open circular ring centered at z_0 and is called an open circular annulus.



Def. Neighborhoods : A set S is said to be a neighborhood of a point z_0 if there exists an open disc centered at z_0 which is entirely contained in S and the point z_0 is said to be an interior point of S.

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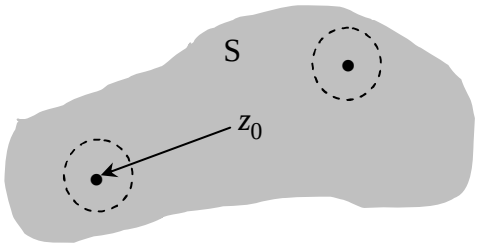
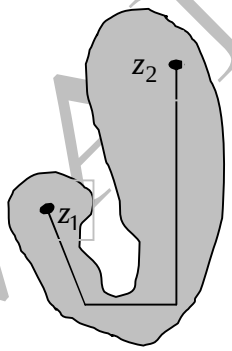
Def. Deleted neighborhood : A set S is said to be a deleted neighborhood of a point z_0 if there exists an open disc centered at z_0 which is entirely contained in S excluding the point z_0.

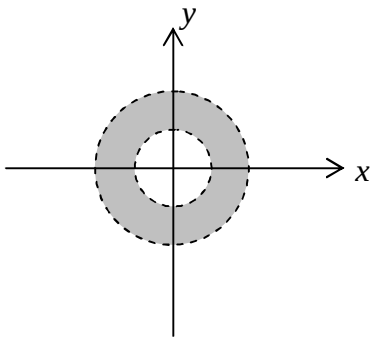
Def. Open sets : A set is said to be an open set if it is neighborhood of each of its points. In other words, if every point z of a set S is an interior point, then S is said to be an open set.



Def. Connected Set : If any pair of points z_1 and z_2 in a set S can be connected by a polygonal line that consists of a finite number of line segements joined end to end that lies entirely in the set, then the set S is said to be connected.



Def. Domain : An open connected set is called a domain.



Def. Regions : A region is a set of points in the complex plane with all, some, or none of its boundary points. Since an open set does not contain any boundary points, it is automatically a region. A region that contains all its boundary points is said to be **closed**. The disc defined by $|z - z_0| \le r$ is an example

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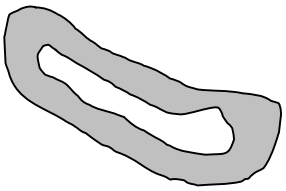
of a closed region and referred to as a **closed disc**. A neighborhood of a point z_0 defined by $|z - z_0| < r$ is an open set or an open region and is said to be an **open disc**.



Def. Bounded Sets : A set *S* in the complex plane is bounded if there exists a real number $R > 0$ such that $|z| < R$ for every *z* in *S*. That is, *S* is bounded if it can be completely enclosed within some neighborhood of the origin.

Some Geometric Loci in complex plane :

1. $\{z \in \mathbb{C} : |z - z_1| = |z - z_2|\}$ represent the perpendicular bisector of line segment joining z_1 and z_2.
2. $\{z \in \mathbb{C} : |z - z_1| + |z - z_2| = K\}$ represent the ellipse with foci z_1 and z_2.

Exercise 1.2

1. Sketch the graph of the given equation in the complex plane :

   (i) $|z - 4 + 3i| = 5$ (ii) $|2z - 1| = 4$ (iii) $\text{Im}(\bar{z} + 3i) = 6$

   (iv) $\text{Im}(z - i) = \text{Re}(z + 4 - 3i)$ (v) $|\text{Re}(1 + i\bar{z})| = 3$ (vi) $z^2 + \bar{z}^2 = 2$

2. Sketch the set *S* of points in the complex plane satisfying the given inequality. Determine whether the set is (a) open, (b) closed, (c) a domain, (d) bounded or (e) connected.

   (i) $2 < \text{Re}(z - 1) < 4$ (ii) $|\text{Re}(z)| > 2$ (iii) $-1 \le \text{Im}(z) < 4$

   (iv) $\text{Re}((2 + i)z + 1) > 0$ (v) $\text{Im}(z) < \text{Re}(z)$ (vi) $\text{Re}(z^2) > 0$

   (vii) $2 \le |z - 3 + 4i| \le 5$ (viii) $1 \le |z - 1 - i| < 2$

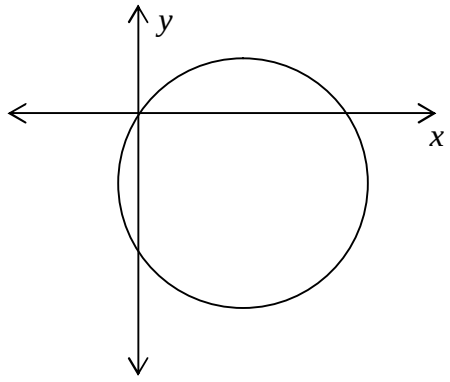
3. Sketch the set of points in the complex plane satisfying the given inequality

   (i) $0 \le \arg(z) \le \frac{\pi}{6}$ (ii) $-\pi < \arg(z) < \frac{\pi}{2}$

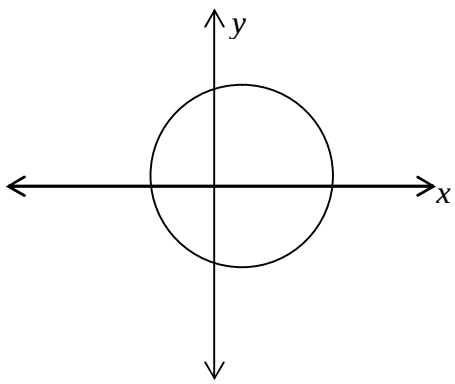
Answers

1.

(i)



(ii)



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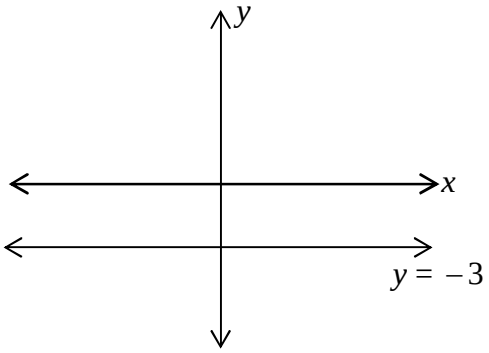
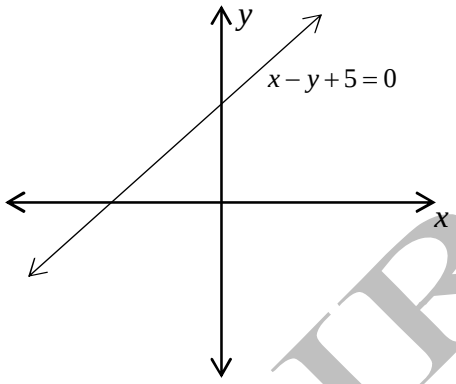
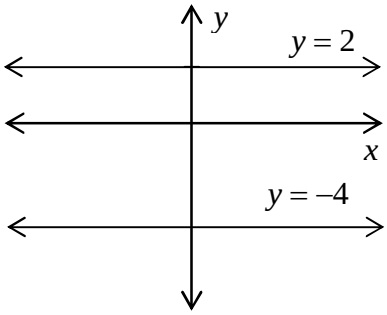
(iii)



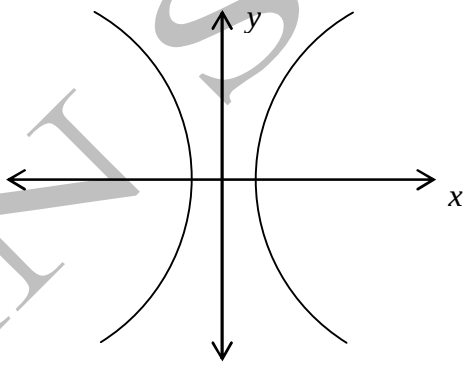
(iv)



(v)

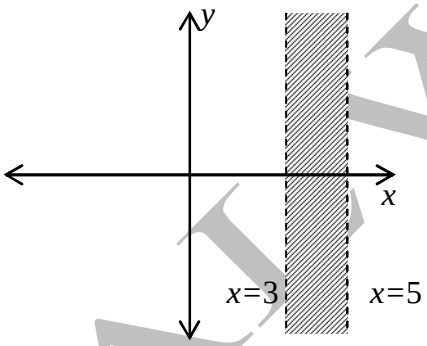


(vi)



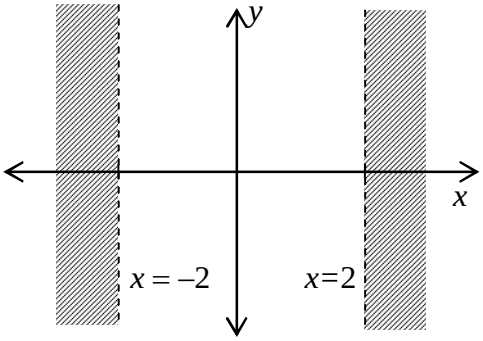
2.

(i)



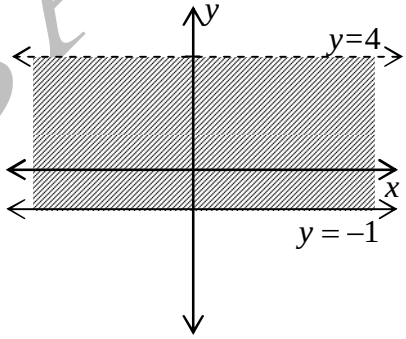
(a) yes (b) no (c) yes (d) no (e) yes

(ii)



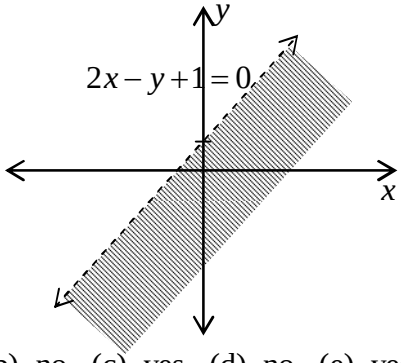
(a) yes (b) no (c) no (d) no (e) no

(iii)



(a) no (b) no (c) no (d) no (e) yes

(iv)



(a) yes (b) no (c) yes (d) no (e) yes

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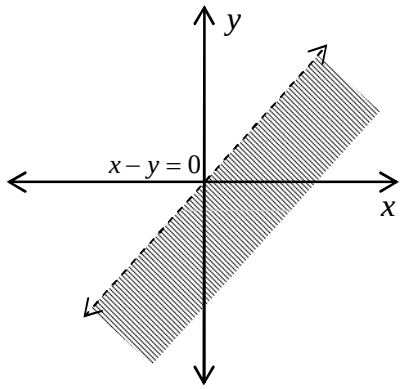
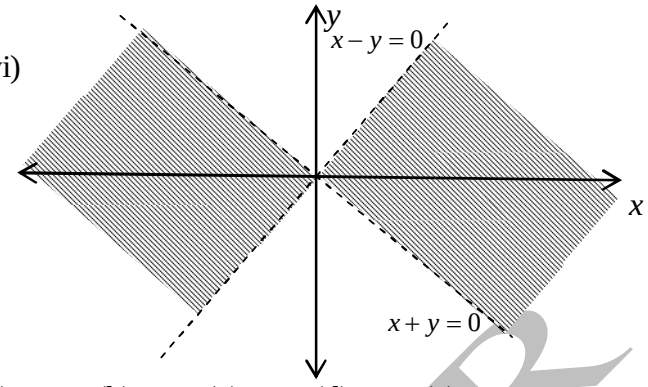
(v)



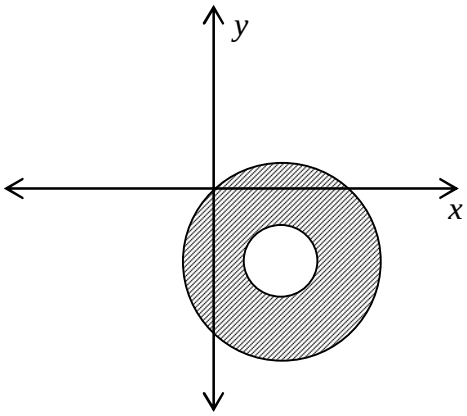
(a) yes (b) no (c) yes (d) no (e) yes

(vi)



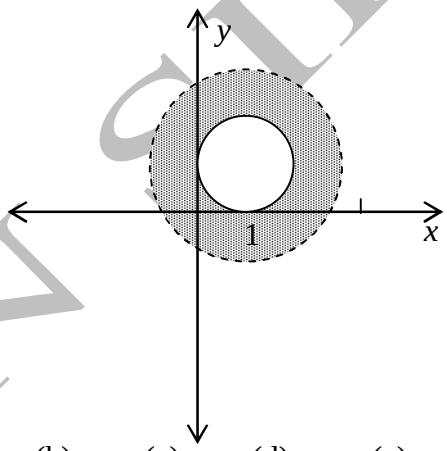
(a) yes (b) no (c) no (d) no (e) no

(vii)



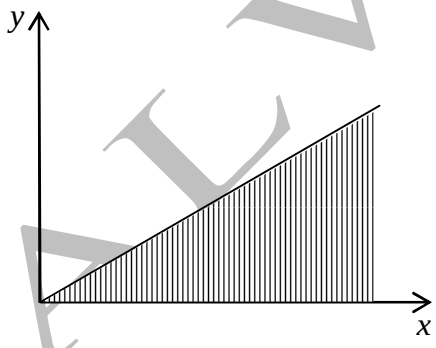
(a) no (b) yes (c) no (d) yes (e) yes

(viii)

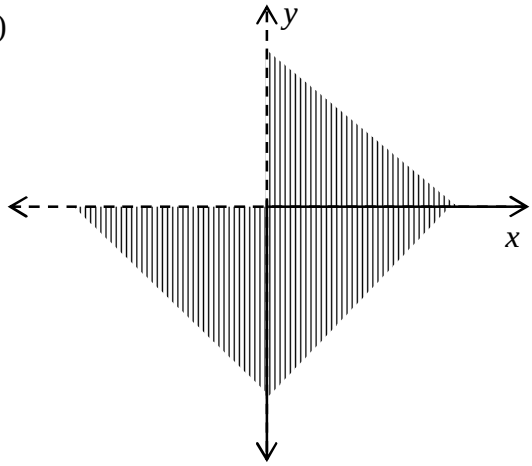


(a) no (b) no (c) no (d) yes (e) yes

3. (i)



(ii)



1.3 Some Elementary Functions

Def. Complex exponential function : The complex exponential function is defined as

$$e^z = e^{x+iy} = e^x\left(\cos y + i\sin y\right)$$

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Results :

1. The modulus of e^z is given by $\left|e^z\right| = e^x$

2. The argument of e^z is given by $\arg e^z = y + 2n\pi$; $n = 0, \pm 1, \pm 2, ...$

3. The conjugate of e^z is given by $\overline{\left(e^z\right)} = e^{\bar{z}}$

4. If z_1 and z_2 are complex numbers then

(i) $e^0 = 1$ (ii) $e^{z_1}.e^{z_2} = e^{z_1+z_2}$ (iii) $e^{z_1} / e^{z_2} = e^{z_1-z_2}$ (iv) $\left(e^{z_1}\right)^n = e^{nz_1}$

5. The complex exponential function e^z is periodic with purely imaginary period of $2\pi i$.

6. For $z = x + iy$,
   (i) $e^z > 0$ when y is an even multiple of π.
   (ii) $e^z < 0$ when y is an odd multiple of π.
   (iii) e^z is purely imaginary when y is an odd multiple of $\frac{\pi}{2}$.
   (iv) $\left|e^{iz}\right| < 1$ when $y > 0$
   (v) $\left|e^{iz}\right| > 1$ when $y < 0$
   (vi) $\left|e^z - 1\right| \le e^{|z|} - 1 \le |z| e^{|z|}$ for any $z \in \mathbb{C}$
   (vii) $\left|\exp(z^2)\right| \le \exp\left(|z|^2\right)$

Def. Complex sine and cosine functions : The complex sine and cosine functions are defined as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Def. The complex tangent, cotangent, secant and cosecant functions are defined as

$$\tan z = \frac{\sin z}{\cos z}, \cot z = \frac{\cos z}{\sin z}, \sec z = \frac{1}{\cos z}, \operatorname{cosec} z = \frac{1}{\sin z}$$

Results :

1. $\sin z = \sin x \cosh y + i \cos x \sinh y$
2. $\cos z = \cos x \cosh y - i \sin x \sinh y$
3. $\left|\sin z\right| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}$
4. $\left|\cos z\right| = \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$
5. $\sin z = 0$ iff $z = n\pi$
6. $\cos z = 0$ iff $z = (2n+1)\frac{\pi}{2}$
7. $\sin z, \cos z, \operatorname{cosec} z$ and $\cot z$ have real periods of 2π.

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8. $\tan z$ and $\cot z$ have real period π.

9. It is important to recognize that some of the properties of real trigonometric functions are not satisfied by their complex counter parts. e.g. $|\sin x| \le 1$ and $|\cos x| \le 1$, but $|\sin z|$ and $|\cos z|$ are unbounded.

10. For $z = x + iy$, $|\sin z|^2 + |\cos z|^2 = 1 \Leftrightarrow z = x$, a real.

Def. Complex hyperbolic functions : The complex hyperbolic sine and hyperbolic cosine functions are defined as $\sinh z = \dfrac{e^z - e^{-z}}{2}$ and $\cosh z = \dfrac{e^z + e^{-z}}{2}$

Def. The complex hyperbolic tangent, cotangent, secant and cosecant functions are defined as

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z}, \qquad \operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{cosech} z = \frac{1}{\sinh z}$$

Results :

1. $\sin z = -i \sinh(iz)$
2. $\cos z = \cosh(iz)$
3. $\sinh z = -i \sin(iz)$
4. $\cosh z = \cos(iz)$
5. $\sinh z = 0$ iff $z = n\pi i$
6. $\cosh z = 0$ iff $z = (2n+1)\dfrac{\pi}{2} i$
7. For $z = x + iy$, $\sinh y \le |\sin(x \pm iy)| \le \cosh y$
8. For $z = x + iy$, $\sinh y \le |\cos(x \pm iy)| \le \cosh y$

Table of formulae :

Sr. No.		Trigonometric Functions	Hyperbolic Functions
1.	(i)	$\sin^2 z + \cos^2 z = 1$	$\cosh^2 z - \sinh^2 z = 1$
	(ii)	$\sec^2 z - \tan^2 z = 1$	$\operatorname{sech}^2 z + \tanh^2 z = 1$
	(iii)	$\operatorname{cosec}^2 z - \cot^2 z = 1$	$\coth^2 z - \operatorname{cosech}^2 z = 1$
2.	(i)	$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$	$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$
	(ii)	$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$	$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$
	(iii)	$\tan(z_1 \pm z_2) = \dfrac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$	$\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$
3.	(i)	$\sin(-z) = -\sin z$	$\sinh(-z) = -\sinh z$
	(ii)	$\cos(-z) = \cos z$	$\cosh(-z) = \cosh z$
	(iii)	$\tan(-z) = -\tan z$	$\tanh(-z) = -\tanh z$
4.	(i)	$\sin 2z = 2 \sin z \cos z$	$\sinh 2z = 2 \sinh z \cosh z$
	(ii)	$\cos 2z = \cos^2 z - \sin^2 z$	$\cosh 2z = \cosh^2 z + \sinh^2 z$

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	(iii)	$\tan 2z = \dfrac{2\tan z}{1-\tan^2 z}$	$\tanh 2z = \dfrac{2\tanh z}{1+\tanh^2 z}$
5.	(i)	$\sin 3z = 3\sin z - 4\sin^3 z$	$\sinh 3z = 3\sinh z + 4\sinh^3 z$
	(ii)	$\cos 3z = 4\cos^3 z - 3\cos z$	$\cosh 3z = 4\cosh^3 z - 3\cosh z$
	(iii)	$\tan 3z = \dfrac{3\tan z - \tan^3 z}{1-3\tan^2 z}$	$\tanh 3z = \dfrac{3\tanh z + \tanh^3 z}{1+3\tanh^2 z}$
6.	(i)	$2\sin A\cos B = \sin(A+B)+\sin(A-B)$	$2\sinh A\cosh B = \sinh(A+B)+\sinh(A-B)$
	(ii)	$2\cos A\sin B = \sin(A+B)-\sin(A-B)$	$2\cosh A\sinh B = \sinh(A+B)-\sinh(A-B)$
	(iii)	$2\cos A\cos B = \cos(A+B)+\cos(A-B)$	$2\cosh A\cosh B = \cosh(A+B)+\cosh(A-B)$
	(iv)	$2\sin A\sin B = \cos(A-B)-\cos(A+B)$	$2\sinh A\sinh B = \cosh(A+B)-\cosh(A-B)$
7.	(i)	$\sin C+\sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$	$\sinh C+\sinh D = 2\sinh\dfrac{C+D}{2}\cosh\dfrac{C-D}{2}$
	(ii)	$\sin C-\sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$	$\sinh C-\sinh D = 2\cosh\dfrac{C+D}{2}\sinh\dfrac{C-D}{2}$
	(iii)	$\cos C+\cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$	$\cosh C+\cosh D = 2\cosh\dfrac{C+D}{2}\cosh\dfrac{C-D}{2}$
	(iv)	$\cos C-\cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$	$\cosh C-\cosh D = 2\sinh\dfrac{C+D}{2}\sinh\dfrac{C-D}{2}$

Def. Complex Logarithmic Function : The complex function $\log z$ is defined as

$$\log z = \log_e |z| + i\arg(z)$$

Properties :

(i) $\log(z_1 z_2) = \log z_1 + \log z_2$ (ii) $\log\left(\dfrac{z_1}{z_2}\right) = \log z_1 - \log z_2$ (iii) $\log z_1^{\,n} = n\log z_1$

Result : Real logarithmic function is defined for positive real numbers only whereas complex logarithmic function is defined for all non-zero complex numbers.

Example : $\log(-5), \log(-7)$ are not defined in real analysis but they are defined in complex analysis. $\log(0)$ is never defined.

Example : Find all complex solutions to each of the following equations :

(a) $e^w = i$ (b) $e^w = 1+i$ (c) $e^w = -2$

Solution : For each equation $e^w = z,$ the set of solutions is given by $w = \log z$ where

$$\log z = \log_e |z| + i\arg(z)$$

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(a) For $z = i$, we have $|z| = 1$ and $\arg(z) = \frac{\pi}{2} + 2n\pi$, we obtain

$$w = \log i = \log_e 1 + i\left(\frac{\pi}{2} + 2n\pi\right) \quad \Rightarrow \quad w = \frac{(4n+1)\pi}{2} i, \qquad n = 0, \pm 1, \pm 2, ...$$

Therefore, each of the values $w = ..., -\frac{7\pi}{2} i, -\frac{3\pi}{2} i, \frac{\pi}{2} i, \frac{5\pi}{2} i, ...$ satisfies the equation $e^w = i$.

(b) For $z = 1 + i$, we have $|z| = \sqrt{2}$ and $\arg(z) = \frac{\pi}{4} + 2n\pi$, we obtain

$$w = \log(1+i) = \log_e \sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)$$

$$\Rightarrow \quad w = \frac{1}{2}\log_e 2 + \frac{(8n+1)\pi}{4} i, \qquad n = 0, \pm 1, \pm 2, ...$$

Each value of w is a solution to $e^w = 1 + i$.

(c) For $z = -2$, we have $|z| = 2$ and $\arg(z) = \pi + 2n\pi$, we obtain

$$w = \log(-2) = \log_e 2 + i(\pi + 2n\pi)$$

$$\Rightarrow \quad w = \log_e 2 + (2n+1)\pi\, i, \qquad n = 0, \pm 1, \pm 2, ...$$

Each value of w is a solution to $e^w = -2$.

Def. Principal value of the complex logarithm : The principal value of the complex logarithm is denoted by $\text{Log}\, z$ and is defined as $\text{Log}\, z = \log_e |z| + i \arg(z)$

Example : Compute the principal value of the complex logarithm

(a) $\text{Log}(i)$ (b) $\text{Log}(1+i)$ (c) $\text{Log}(-2)$

Solution : (a) For $z = i$, we have $|z| = 1$ and $\text{Arg}(z) = \frac{\pi}{2}$ and so

$$\text{Log}\, i = \log_e 1 + \frac{\pi}{2} i \quad \Rightarrow \quad \text{Log}\, i = \frac{\pi}{2} i$$

(b) For $z = 1 + i$, we have $|z| = \sqrt{2}$ and $\text{Arg}(z) = \frac{\pi}{4}$ and so

$$\text{Log}(1+i) = \log_e \sqrt{2} + \frac{\pi}{4} i$$

$$\Rightarrow \quad \text{Log}(1+i) = \frac{1}{2}\log_e 2 + \frac{\pi}{4} i$$

(c) For $z = -2$, we have $|z| = 2$ and $\text{Arg}(z) = \pi$ and so

$$\text{Log}(-2) = \log_e 2 + \pi i$$

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PERIODIC TABLE

Sr. No.	Complex function	Periodic	Period	Real function	Periodic	Period
1.	e^z	Yes	$2\pi i$	e^x	No	---
2.	$\sin z$	Yes	2π	$\sin x$	Yes	2π
3.	$\cos z$	Yes	2π	$\cos x$	Yes	2π
4.	$\sinh z$	Yes	$2\pi i$	$\sinh x$	No	---
5.	$\cosh z$	Yes	$2\pi i$	$\cosh x$	No	---

Exercise 1.3

1. Find the real part, imaginary part, modulus and conjugate of the following functions :

   (i) $f(z)=e^z$ (ii) $f(z)=e^{-iz}$ (iii) $f(z)=e^{z^2}$ (iv) $f(z)=\sin z$

   (v) $f(z)=\cos z$ (vi) $f(z)=\sin h\, z$ (vii) $f(z)=\cosh z$

2. Solve the following equations :

   (i) $e^z=0$ (ii) $e^z=1$ (iii) $e^z=-1$ (iv) $\sin z=0$

   (v) $\sin z=1$ (vi) $\sin z=-1$ (vii) $\cos z=0$ (viii) $\cos z=1$

   (ix) $\cos z=-1$ (x) $\sinh z=0$ (xi) $\cosh z=0$

3. Show that the complex exponential function, complex trigonometric functions and the complex hyperbolic functions are periodic. Also find their periods.
4. How many real counter parts of above functions are periodic. Find the period wherever applicable.
5. Write the given expression in terms of x and y

   (i) $\left|e^{z^2-z}\right|$ (ii) $\arg\left(e^{i(z+\bar{z})}\right)$ (iii) $\overline{ie^z+1}$

6. Find all complex values of the given logarithm

   (i) $\log(-5)$ (ii) $\log(-2+2i)$ (iii) $\log\left(\sqrt{2}+\sqrt{6}\,i\right)$

   (iv) $\log(-ei)$ (v) $\log(1+i)$ (vi) $\log\left(-\sqrt{3}+i\right)$

7. Write the principal value of the logarithm in the form $a+ib$

   (i) $\text{Log}(6-6i)$ (ii) $\text{Log}\left(-e^2\right)$ (iii) $\text{Log}\left[\left(1+\sqrt{3}\,i\right)^5\right]$ (iv) $\text{Log}\left[(1+i)^4\right]$

Answers

1. (i) $e^x\cos y; e^x\sin y; e^x; e^{\bar{z}}$ (ii) $e^y\cos x; -e^y\sin x; e^y; e^{i\bar{z}}$

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(iii) $e^{x^2-y^2}\cos(2xy); e^{x^2-y^2}\sin(2xy); e^{x^2-y^2}; e^{\bar{z}^2}$

(iv) $\sin x\cosh y; \cos x\sinh y; \sqrt{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y}; \sin\bar{z}$

(v) $\cos x\cosh y; -\sin x\sinh y; \sqrt{\cos^2 x\cosh^2 y+\sin^2 x\sinh^2 y}; \cos\bar{z}$

(vi) $\sinh x\cos y; \cosh x\sin y; \sqrt{\sinh^2 x\cos^2 y+\cosh^2 x\sin^2 y}; \sinh\bar{z}$

(vii) $\cosh x\cos y; \sinh x\sin y; \sqrt{\cosh^2 x\cos^2 y+\sinh^2 x\sin^2 y}; \cosh\bar{z}$

2. (i) Such a z does not exist. (ii) $z=2n\pi i,\ n\in\mathbb{Z}$ (iii) $z=(2n+1)\pi i,\ n\in\mathbb{Z}$

(iv) $z=n\pi,\ n\in\mathbb{Z}$ (v) $z=(4n+1)\frac{\pi}{2},\ n\in\mathbb{Z}$ (vi) $(4n-1)\frac{\pi}{2},\ n\in\mathbb{Z}$

(vii) $z=(2n+1)\frac{\pi}{2},\ n\in\mathbb{Z}$ (viii) $z=2n\pi,\ n\in\mathbb{Z}$ (ix) $z=(2n+1)\pi,\ n\in\mathbb{Z}$

(x) $z=n\pi i,\ n\in\mathbb{Z}$ (xi) $z=(2n+1)\frac{\pi}{2}i,\ n\in\mathbb{Z}$

5. (i) $e^{x^2-x-y^2}$ (ii) $2x+2n\pi,\ n=0,\pm1,\pm2,...$ (iii) $1-e^x(\sin y+i\cos y)$

6. (i) $\log_e 5+(2n+1)\pi i$ (ii) $\frac{3}{2}\log_e 2+\frac{(8n+3)}{4}\pi i$ (iii) $\frac{3}{2}\log_e 2+\frac{(6n+1)}{3}\pi i$

(iv) $1+\frac{1}{2}(4n+3)\pi i$ (v) $\frac{1}{2}\log_e 2+\frac{(8n+1)}{4}\pi i$ (vi) $\log_e 2+\frac{(12n+5)}{6}\pi i$

7. (i) $\frac{1}{2}\log_e 72-\frac{1}{4}\pi i$ (ii) $2+\pi i$ (iii) $5\log_e 2-\frac{1}{3}\pi i$ (iv) $\log_e 4+\pi i$

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True-false exercise

1. $\mathrm{Re}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Re}(z_2)$.

2. $|z-1| = |\overline{z}-1|$.

3. If $\mathrm{Im}(z) > 0$, then $\mathrm{Re}(1/z) > 0$.

4. $i < 10i$

5. If $z \neq 0$, then $\mathrm{Arg}(z+\overline{z}) = 0$.

6. $|x+iy| \leq |x| + |y|$

7. $\arg(\overline{z}) = \arg\left(\frac{1}{z}\right)$

8. $\arg(-2+10i) = \pi - \tan^{-1}(5) + 2n\pi$ for $n \in \mathbb{Z}$.

9. If z is a root of a polynomial equation $a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0$, then $\overline{z}$ is also a root.

10. For any nonzero complex number z, there are an infinite number of values for $\arg(z)$.

11. If $|z-2| < 2$, then $|\mathrm{Arg}(z)| < \frac{\pi}{2}$.

12. The equation $z^n = 1$, n a positive integer, will have only real solutions for $n = 1$ and $n = 2$.

13. If $f(z)$ is a complex function, then $f(x+0i)$ must be a real number.

14. $\mathrm{Re}\, z^{1/2} > 0$ for all $z \in \mathbb{C} \setminus \{0\}$.

15. The domain of the function $f(z) = \frac{1}{z^2 + i}$ is all complex numbers.

16. The domain of the function $f(z) = e^{z^2 - (1+i)z + 2}$ is all complex numbers.

17. If $f(z)$ is a complex function with $u(x, y) = 0$, then the range of f lies in the imaginary axis.

18. The range of the function $f(z) = \mathrm{Arg}(z)$ is all real numbers.

19. There are no solutions to the equation $\mathrm{Arg}(z) = \mathrm{Arg}(z^3)$.

20. There are infinitely many solutions z to the equation $e^z = w$.

21. $\ln i = \frac{\pi i}{2}$.

22. $\mathrm{Im}(\ln z) = \arg(z)$.

23. Ln z is an entire function.

24. $\cos z$ is a periodic function with a period of 2π .

25. The set of points z such that $|z+bi| < |z+b|$, $b > 0$ is the half-plane $\{z = x+iy : y < x\}$

26. The inequality $|z_1 - z_2| < |\overline{z}_1 + z_2|$ holds provided either $\mathrm{Re}\, z_1 > 0$ and $\mathrm{Re}\, z_2 > 0$, or $\mathrm{Re}\, z_1 < 0$ and $\mathrm{Re}\, z_2 < 0$.

27. Two complex numbers z_1 and z_2 whose sum and difference are real and purely imaginary, respectively, must satisfy $z_2 = \overline{z}_1$.

28. The roots of the equation $\tan z + \cot z = 2$ are at $z = (4k+1)\pi/4$, $k \in \mathbb{Z}$.

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29. If the sum and product of two complex numbers are both real then either both the complex numbers are real or one is conjugate of the other.

30. The inequalities $\operatorname{Re} z > 0$ and $|z-1| < |z+1|$ are equivalent.

31. The inequalities $|z| < 1$ and $\operatorname{Re}\left(\frac{1+z}{1-z}\right) > 0$ are equivalent.

32. If f is any one of the six hyperbolic (circular) functions, then, for all $z \in \mathbb{C}$, $\overline{f(z)} = f(\overline{z})$.

33. For $\operatorname{Re} z_j > 0 \ \ (j = 1, 2)$, $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$

34. The complex roots of a quadratic equation have the property that one is the square of the other.

35. If $|z| < 1$, then both $\operatorname{Re}(1+z) > 0$ and $\operatorname{Re}(1+z)^2 > 0$ do not hold simultaneously.

36. If $z^2 = (\overline{z})^2$, then z is either real or purely imaginary.

37. The equation $\frac{1+\cos\phi+i\sin\phi}{1-\cos\phi+i\sin\phi} = e^{i\left(\phi-\frac{\pi}{2}\right)}\cot\left(\frac{\phi}{2}\right)$ holds for each $\phi \neq 2n\pi$, $n \in \mathbb{Z}$.

38. The non real roots of $(1+z)^4 = 16z^4$ are $(-1 \pm 2i)/5$.

39. The product of the distinct n-th roots of unity is $(-1)^{n-1}$.

40. All the solutions of $z^4 + 81 = 0$ are $3[\pm 1 \pm i]/\sqrt{2}$.

41. The set of complex numbers z such that $\arg(z-i) = \frac{\pi}{3}$ represents the equation of the straight line $y = \sqrt{3}x + 1$.

42. If $\operatorname{Arg}(z+3) = \frac{\pi}{3}$, then the least value of $|z|$ is $3\sqrt{3}/2$.

43. If $|z-(4-3i)| = 2$, then the greatest and least value of $|z|$ are 7 and 3 respectively.

44. $|\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|$ for every complex number z.

45. $f(z) = \arg(z)$ is a single valued function on $[t, t+2\pi]$ for any real t.

46. $\operatorname{Arg}\left(\frac{1}{z}\right) = -\operatorname{Arg}(z)$ iff z doesn't lies on negative real axis.

47. $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z)$ for all $z \in \mathbb{C}$

48. $\arg(z) + \arg(\overline{z}) = 2n\pi$

49. $\arg(z) - \arg(-z) = -\pi + 2n\pi$

50. Let α be n^{th} primitive root of unity and ω be any root of equation $z^n = 1$ then all the roots of equation $z^n = 1$ are given by $\omega, \omega\alpha, \omega\alpha^2, \omega\alpha^3, ..., \omega\alpha^{n-1}$

Assignment

---------- S C Q ----------

1. If $z = x + iy$, then the number of solutions of the equation $z^2 = \overline{z}$ is
   1. one
   2. two
   3. four
   4. infinite

2. If $\frac{4+3i}{3-4i} = x + iy$, then $\frac{x}{y}$ is equal to
   1. 0
   2. 1
   3. $\frac{4}{3}$
   4. $\frac{4}{5}$

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3. If $1, \omega, \omega^2$ are the cube roots of unity, then $(x-y)(x-\omega y)(x-\omega^2 y)$ is equal to
   1. $x-y$ 2. x^2-y^2
   3. x^3-y^3 4. x^3+y^3

4. The real part of the complex number $(1+i)^n$ is
   1. $2^{n/2}\cos\frac{n\pi}{4}$ 2. $2^n\cos\frac{n\pi}{2}$
   3. $2^{-n/2}\cos n\pi$ 4. $2^{-n}\cos\frac{n\pi}{2}$

5. If $x=-2-\sqrt{3}i$, then the value of $2x^4+5x^3+7x^2+41$ is
   1. $4+\sqrt{3}i$ 2. $4-\sqrt{3}i$
   3. $\sqrt{3}+4i$ 4. $\sqrt{3}-4i$

6. If $|z|=|z-1|$, then
   1. $\mathrm{Re}(z)=1$ 2. $\mathrm{Re}(z)=\frac{1}{2}$
   3. $\mathrm{Im}(z)=1$ 4. $\mathrm{Im}(z)=\frac{1}{2}$

7. If $\left|\frac{z-5i}{z+5i}\right|=1$, then $z=x+iy$ lies on
   1. the real axis
   2. the straight line $x=5$
   3. the straight line $y=5$
   4. a circle passing through origin

8. If the imaginary part of $\frac{2z+1}{iz+1}$ is -2, then the locus of a point representing z, is a
   1. circle 2. straight line
   3. parabola 4. None of these

9. For complex number z, $|z+5|^2+|z-5|^2=75$ represents
   1. a circle 2. an ellipse
   3. a triangle 4. a straight line

10. The correct polar form of the complex number $1-i$ is
    1. $\sqrt{2}e^{\frac{\pi}{4}i}$ 2. $e^{-\frac{\pi}{4}i}$
    3. $\sqrt{2}e^{-\frac{\pi}{4}i}$ 4. $e^{\frac{\pi}{4}i}$

11. If $1, \omega, \omega^2$ are the cube root of unity, then the roots of $(x-1)^3+8=0$ are
    1. $-1,-1,-1$ 2. $-1,1+2\omega,1+2\omega^2$
    3. $1,\omega,2\omega$ 4. $-1,1-2\omega,1-2\omega^2$

12. If $|z-1|=2$, then the value of $z\bar{z}-z-\bar{z}$ is
    1. 4 2. 2
    3. 1 4. 3

13. The solution of the equation $|z|-z=1+2i$ is
    1. $1-2i$ 2. $2-\frac{3}{2}i$
    3. $\frac{3}{2}+2i$ 4. $\frac{3}{2}-2i$

14. The complex numbers $z_1=1+2i$, $z_2=4-2i$ and $z_3=1-6i$ form the vertices of a
    1. right angled triangled
    2. isosceles triangle
    3. equilateral triangle
    4. scalene triangle

15. An annulus $\rho_1<|z-a|<\rho_2$ is
    1. connected 2. disconnected
    3. semi-connected 4. None of these

16. The complement of the unit circle $|z|=1$ is
    1. $|z|>1$ 2. $|z|<1$
    3. $|z|=1$ 4. None of these

17. The set S is open, then which of the following is true?
    1. S does not contains its boundary points
    2. S contains its boundary points
    3. S have its boundary points

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4. None of the above

18. The set S is closed if
1. it does not contains its boundary points
2. it has no boundary points
3. it contains its boundary points
4. None of the above

19. If z_1, z_2 are two complex numbers such that $|z_1+z_2|=|z_1|+|z_2|$ Then, it is necessary that
1. $z_1=z_2$
2. $z_2=0$
3. $z_1=\lambda z_2$ for some real number λ
4. $z_1z_2=0$ or $z_1=\lambda z_2$ for some real number λ

20. If z is a complex number such that $\mathrm{Re}(z)\geq 0$. Let $\mathrm{Arg}(z)$ represent the principal argument of z, then the interval in which $\mathrm{Arg}(z)$ lies is
1. $[0,\pi]$
2. $[-\pi,0]$
3. $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$
4. None of these

21. The principal argument of the complex number $\frac{2+i}{4i+(1+i)^2}$ is
1. $\tan^{-1}(-2)$
2. $\tan^{-1}\left(\frac{1}{2}\right)$
3. $\tan^{-1}(2)$
4. None of these

22. Let ω be n^{th} root of unity distinct from 1, then the value of $1+2\omega+3\omega^2+....+n\omega^{n-1}$
1. $\frac{n}{1-\omega}$
2. $\frac{n}{\omega-1}$
3. $\frac{n}{\omega^2-1}$
4. $\frac{n^2}{\omega-1}$

23. Let $1,\omega,\omega^2,....,\omega^{n-1}$ be n, n^{th} roots of unity then value of $(a-\omega)(a-\omega^2)(a-\omega^3)....(a-\omega^{n-1})$ is
1. n
2. $\frac{a^2-1}{8}$
3. 0
4. $\frac{a^n-1}{8}$

24. Let $z_1\in\mathbb{C}$ be any fixed complex number then the equation $|z-z_1|=|z-\overline{z}_1|$ represents
1. unit circle with center origion
2. Real-axis
3. Imaginary axis
4. None of these

25. The number $\sqrt{2}\,e^{i\pi}$ is
1. an irrational number
2. a transcendental number
3. a rational number
4. imaginary number

26. If z_1 and z_2 are distinct complex numbers such that $|z_1|=|z_2|=1$ and $z_1+z_2=1$, then the triangle in the complex plane with z_1, z_2 and -1 as vertices
1. must be equilateral
2. must be right angle
3. must be isosceles, but not necessarily equilateral
4. must be obtuse angled

(CSIR NET June 2013)

27. $f(z):\mathbb{C}\to\mathbb{C}$ defined by $f(z)=\sin z$ is
1. one-one
2. onto
3. one-one, onto both
4. neither one-one nor onto

(GATE 1995)

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28. The real part of the principal value of 4^{4-i} is
    1. $256\cos(\ln 4)$ 2. $64\cos(\ln 4)$
    3. $16\cos(\ln 4)$ 4. $4\cos(\ln 4)$

(GATE 2004)

---------- **M C Q** ----------

1. If $i^{i^{i^{\cdots\infty}}} = \alpha + i\beta$, then which of the following is /are true ?
    1. $\frac{\alpha}{\beta} = \tan\frac{\pi\alpha}{2}$
    2. $\alpha^2 + \beta^2 = e^{-\pi\beta}$
    3. $\alpha^2 + \beta^2 = e^{\pi\beta}$
    4. $\frac{\beta}{\alpha} = \tan\frac{\pi\alpha}{2}$

2. Let ω be primitive n^{th} root of unity then the value of $1+\omega^k+\omega^{2k}+\ldots+\omega^{(n-1)k}$; $k \in \mathbb{N}$
    1. n if k is multiple of n
    2. 0 if k is multiple of n
    3. n if k is not multiple of n
    4. 0 if k is not multiple of n

3. Let $z_1, z_2 \in \mathbb{C}$ then the equation $|z-z_1|+|z-z_2| = |z_1-z_2|$ represent
    1. line segment joining z_1 and z_2
    2. line passing through z_1 and z_2
    3. unit circle with center at origin
    4. real-axis

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and let u be the real part of f and v the imaginary part of f. Then, for $x, y \in \mathbb{R}$, $|f'(x+iy)|^2$ is equal to
    1. $u_x^2 + u_y^2$ 2. $u_x^2 + v_x^2$
    3. $v_y^2 + u_y^2$ 4. $v_y^2 + v_x^2$

(CSIR NET Dec 2017)

True false key

1. F	2. T	3. F	4. F
5. T	6. T	7. T	8. T
9. F	10. T	11. T	12. T
13. F	14. F	15. F	16. T
17. T	18. T	19. F	20. T
21. F	22. T	23. F	24. T
25. T	26. T	27. T	28. T
29. T	30. T	31. T	32. T
33. T	34. F	35. F	36. T
37. T	38. T	39. T	40. T
41. T	42. T	43. T	44. T
45. F	46. T	47. F	48. T
49. T	50. T		

Assignment key

S C Q

1. 3	2. 1	3. 3	4. 1
5. 2	6. 2	7. 1	8. 2
9. 1	10. 3	11. 4	12. 4
13. 4	14. 2	15. 1	16. 4
17. 1	18. 3	19. 4	20. 3
21. 1	22. 2	23. 4	24. 2
25. 1	26. 1	27. 1	28. 2

M C Q

1. 2,4 2. 1,4 3. –
4. 1,2,3,4

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Chapter - 2

Analytic Functions

2.1 Analytic Functions

Def. Limit of a complex function : Suppose that a complex function f is defined in a deleted neighborhood of z_0 and suppose that L is a complex number. The limit of f as z tends to z_0 exists and is equal to L , written as $\lim_{z\to z_0} f(z) = L$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Criterion for the non-existence of a limit : If f approaches two complex numbers $L_1 \neq L_2$ for two different curves or paths through z_0 , then $\lim_{z\to z_0} f(z)$ does not exist.

Def. Continuity of a complex function: A complex function f is continuous at a point z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

Criteria for continuity at a point: A complex function f is continuous at a point z_0 if each of the following three conditions hold :

(i) $\lim_{z\to z_0} f(z)$ exists,

(ii) f is defined at z_0 , and

(iii) $\lim_{z\to z_0} f(z) = f(z_0)$

Result 1 : A function $f(x) = u(x,y) + iv(x,y)$ is continuous at a point $z_0 = x_0 + iy_0$ iff $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y). In other words, a function is continuous iff its real and imaginary parts are continuous.

Def. Differentiable function : The function $f(z) = u(x,y) + iv(x,y)$ is said to be differentiable at a point $z = z_0$ if $f(z)$ is continuous at $z = z_0$ and its limit *i.e* $f'(z) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

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Result 2 : A necessary condition for differentiability : If a function $f(z)=u(x,y)+iv(x,y)$ is differentiable at a point $z=x+iy$, then the first order partial derivatives of *u* and *v* exist and satisfy the Cauchy-Riemann equations i.e. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Result 3 : If a function is differentiable at a point then C-R equations hold at that point.

OR

If C-R equations does not hold at a point then function can not be differentiable at that point.

Result 4 : Sufficient conditions for differentiability : If the real functions $u(x,y)$ and $v(x,y)$ are continuous and have continuous first-order partial derivatives in some neighborhood of a point *z*, and if $u(x,y)$ and $v(x,y)$ satisfy the C-R equations at *z*, then the complex function $f(z)=u(x,y)+iv(x,y)$ is differentiable at *z* and $f'(z)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

Result 5 : If a function $f(z)=u(x,y)+iv(x,y)$ is differentiable at a point *z* then its derivative is given by $f'(z)=u_x+iv_x$ or $f'(z)=v_y-iu_y$

Result 6 : Different forms of C-R equations : Following are the different forms of C-R equations :

(i) If $f(z)=u(x,y)+iv(x,y)$ then C-R equations are $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

(ii) $\frac{\partial f}{\partial \bar{z}}=0$, where $\bar{z}=x-iy$

(iii) $i\,f_x(x,y)=f_y(x,y)$; where $f_x(x,y)$ represents the partial derivative of f w.r.t. *x* and $f_y(x,y)$ w.r.t. *y*.

Result 7 : Polar form of the C-R equations is $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Def. Rational function : A function of the form $\frac{p(z)}{q(z)}$, $q(z)\neq 0$, where $p(z)$ and $q(z)$ are the polynomials is called a rational function.

Def. Analytic function : A function $f(z)=u+iv$ is analytic at a point *z*, if $f(z)$ is differentiable at *z* as well as in some neighbourhood of the point *z*.

Def. Holomorphic Function : A function that is analytic throughout a domain *D* is called holomorphic function or regular function.

Def. Entire function : If a function $f(z)$ is analytic in the whole complex plane then that function is called the entire function.

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Result 8 : If a function is analytic in a domain *D*, then it is infinitely many times differentiable in *D*.

Remark : This result is a sharp contrast with its real counter part.

Result 9 : Set of all analytic points form an open set.

Result 10 : If a function is analytic at some point than it is analytic at uncountable number of points.

Result 11 : If a function is analytic at some point than it is differentiable at uncountable number of points.

Example 1 : Show that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{-i\,\partial w}{\partial y}$ where $w = u + iv$ is an analytic function.

Solution : By Cauchy – Riemann equations, we have

$$\frac{dw}{dz} = u_x + i v_x = v_y - i u_y \qquad \text{.....(1)}$$

Now, since $w = u + iv$ $\Rightarrow$ $\frac{\partial w}{\partial x} = u_x + i v_x$ .....(2)

By (1) and (2) , we obtain $\frac{dw}{dz} = \frac{\partial w}{\partial x}$

Further , $\frac{\partial w}{\partial y} = u_y + i v_y$

$$\Rightarrow \quad -i\frac{\partial w}{\partial y} = v_y - i u_y \qquad \text{.....(3)}$$

By (1) and (3), we have $\frac{dw}{dz} = -i\frac{\partial w}{\partial y}$

Thus , we have $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$

Example 2 : If $p(z)$ is a non-constant polynomial than $f(z) = \overline{p(z)}$ is no-where analytic.

Proof : $f(z) = \overline{p(z)} = \overline{a_n}\,\overline{z}_0^{\,n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \ldots + \overline{a_0}$

$$\frac{\partial f}{\partial z} = n\overline{a_n}\, z_0^{n-1} + (n-1)\overline{a_{n-1}}\,\overline{z}^{n-2} + \ldots + \overline{a_1}$$

$\frac{\partial f}{\partial z} = 0$ at only finite number of points

Hence $p(\overline{z})$ is no-where analytic.

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Results :

1. $\frac{d}{dz}(\sin z) = \cos z$
2. $\frac{d}{dz}(\cos z) = -\sin z$
3. $\frac{d}{dz}(\tan z) = \sec^2 z$
4. $\frac{d}{dz}(\cot z) = -\text{cosec}^2 z$
5. $\frac{d}{dz}(\sec z) = \sec z \tan z$
6. $\frac{d}{dz}(\text{cosec}\, z) = -\text{cosec}\, z \cot z$
7. $\frac{d}{dz}(\sinh z) = \cosh z$
8. $\frac{d}{dz}(\cosh z) = \sinh z$
9. $\frac{d}{dz}(\tanh z) = \text{sech}^2 z$
10. $\frac{d}{dz}(\coth z) = -\text{cosech}^2 z$
11. $\frac{d}{dz}(\text{sech}\, z) = -\text{sech}\, z \tanh z$
12. $\frac{d}{dz}(\text{cosech}\, z) = -\text{cosech}\, z \coth z$

Exercise 2.1

1. Determine the points on which the following functions are continuous, differentiable and analytic :

   (i) $f(z) = c$ (ii) $f(z) =$ polynomial function (iii) $f(z) = \sin z$

   (iv) $f(z) = \cos z$ (v) $f(z) = e^z$ (vi) $f(z) = \sin h\, z$ (vii) $f(z) = \cosh z$

2. Determine the points on which the following functions are continuous, differentiable and analytic :

   (i) $f(z) = \tan z$ (ii) $f(z) = \sec z$ (iii) $f(z) = \cot z$ (iv) $f(z) = \text{cosec}\, z$

   (v) $f(z) = \tanh z$ (vi) $f(z) = \text{sec}\, h\, z$ (vii) $f(z) = \coth z$ (viii) $f(z) = \text{cosech}\, z$

3. Determine the points on which the following functions are differentiable :

   (i) $f(z) = \text{Re}(z)$ (ii) $f(z) = \text{Im}(z)$ (iii) $f(z) = \bar{z}$

   (iv) $f(z) = |z|$ (v) $f(z) = |z|^2$ (vi) $f(z) = (2x^2 + y) + i(y^2 - x)$

   (vii) $f(z)$ is Rational function (viii) $f(z) = \text{Arg}(z)$ (ix) $f(z) = \text{Log}\, z$

   (x) $f(z) = \text{Log}\,(z^2 - 1)$

Answers

1. continuous,differentiable and analytic everywhere in all parts
2. (i),(ii) continuous,differentiable and analytic everywhere except at $z = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$

   (iii),(iv) continuous,differentiable and analytic everywhereb except at $z = n\pi,\ n \in \mathbb{Z}$

   (v),(vi) continuous,differentiable and analytic everywhere except at $z = (2n+1)\frac{\pi i}{2},\ n \in \mathbb{Z}$

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(vii),(viii) continuous ,differentiable and analytic everywhere except at $z = n\pi i,\ n \in \mathbb{Z}$

3. (i) continuous everywhere but nowhere differentiable and analytic.

(ii) continuous everywhere but nowhere differentiable and analytic.

(iii) continuous everywhere but nowhere differentiable and analytic.

(iv) continuous everywhere but nowhere differentiable and analytic.

(v) continuous everywhere , differentiable only at (0,0), nowhere analytic.

(vi) continuous everywhere, differentiable on the line $y = 2x$, nowhere analytic.

(vii) continuous,differentiable and analytic everywhere except for finitely many points where the denominator is zero.

(viii) continuous, differentiable and analytic everywhere except at non-positive real axis.

(ix) continuous, differentiable and analytic everywhere except at non-positive real axis.

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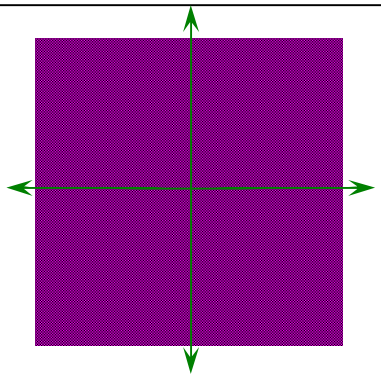
1.



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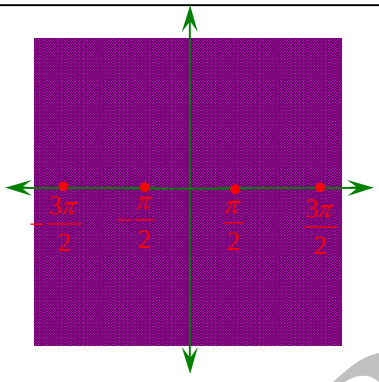

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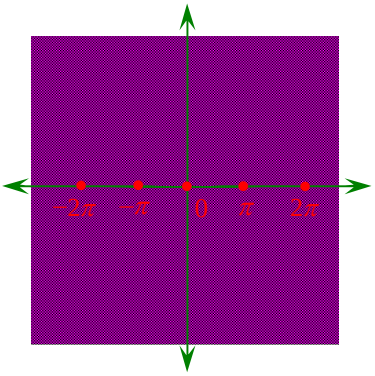

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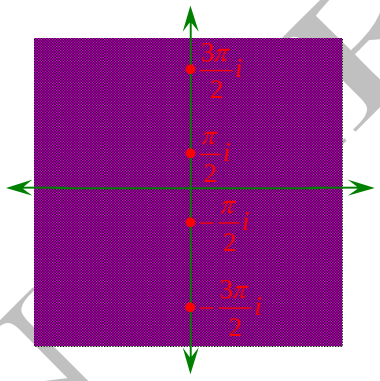

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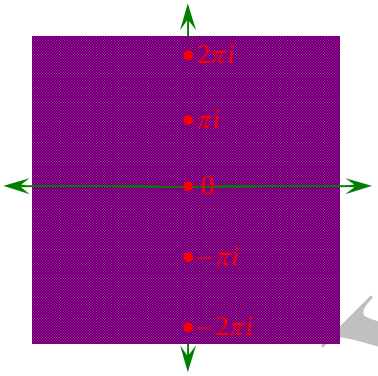

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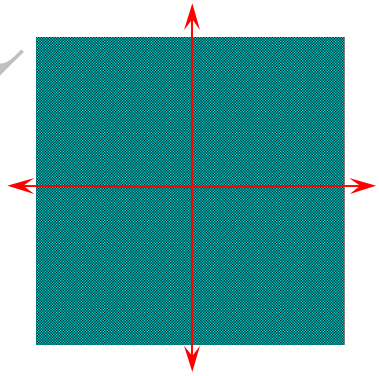

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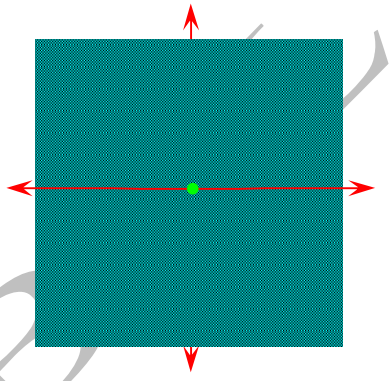

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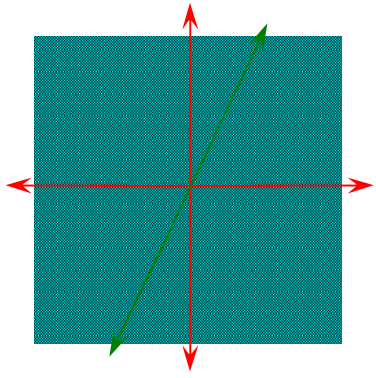

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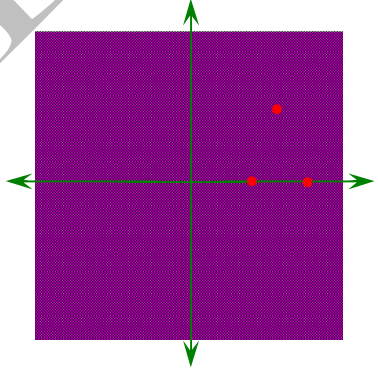

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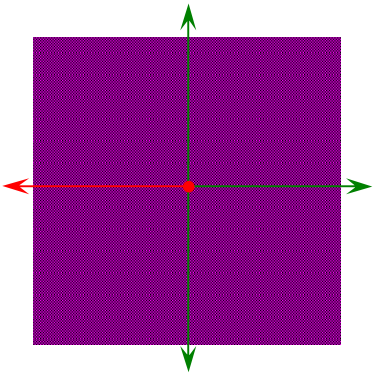

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2.2 Harmonic Functions

Def. Harmonic function : A real function $\phi(x,y)$ of two variables *x* and *y* is said to be Harmonic if it satisfy the Laplace equation i.e. $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ i.e. $\nabla^2\phi=0$ and second order partial derivatives of ϕ are continuous.

Result : The real and imaginary part of an analytic function are harmonic.

Contrapositive : Let $f(z)=u+iv$ be any function s.t. *u* or *v* is not harmonic then function is not analytic.

Def. Harmonic conjugate : Let $u(x,y)$ and $v(x,y)$ are harmonic functions, then $v(x,y)$ is said to be Harmonic Conjugate of $u(x,y)$, if the function $f(z)=u(x,y)+iv(x,y)$ is analytic.

Note: Harmonic conjugacy is not a symmetric property.

Methods to construct harmonic conjugates :

1. Exact differential method :

Case (i) : When *u* is given and *v* is to be constructed :

(a) Verify that *u* is harmonic.

(b) Let $v=v(x,y)$ then $dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy \quad =-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$

This differential equation will be exact always as here $M=-\frac{\partial u}{\partial y}$ and $N=\frac{\partial u}{\partial x}$ so we get

$$\frac{\partial M}{\partial y}=-\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial N}{\partial x}=-\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0 \qquad \left\{\text{Here } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ because } u(x,y) \text{ is Harmonic.}\right\}$$

Then $v=\int\limits_{y \text{ constant}} M\ dx+\int(\text{terms in } N \text{ not containing } x)\,dy+C$

Case (ii) : When *v* is given and *u* is to be constructed :

(a) Verify that *v* is harmonic.

(b) Let $u=u(x,y)$ then $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy \quad =\frac{\partial v}{\partial y}dx-\frac{\partial v}{\partial x}dy$

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This differential equation will be exact always as here $M=\frac{\partial v}{\partial y}$ and $N=-\frac{\partial v}{\partial x}$ so we get

$$\frac{\partial M}{\partial y}=\frac{\partial^2 v}{\partial y^2} \text{ and } \frac{\partial N}{\partial x}=-\frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0 \qquad \left\{\text{Here } \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \text{ because } v(x,y) \text{ is Harmonic.}\right\}$$

Then $u=\int\limits_{y \text{ constant}} M\ dx+\int(\text{terms in } N \text{ not containing } x)\,dy+C$

2. Milne's Thompson Method :

Case(i): If u is given : $f(z)=\int\left[u_x(z,0)-iu_y(z,0)\right]dz+C$

Case(ii): If v is given : $f(z)=\int\left[v_y(z,0)+iv_x(z,0)\right]dz+C$

Case(iii): If $u-v$ is given :

(a) we have $f(z)=u+iv$ .....(1)

$$if(z)=iu-v \qquad \text{.....(2)}$$

Adding (1) and (2), we have

$(1+i)f(z)=(u-v)+i(u+v)$ or

$F(z)=U+iV$, where $F(z)=(1+i)f(z)$, $U=u-v$, $V=u+v$

Now the problem reduces to construct an analytic function $F(z)$ whose real part is given.

(b) Proceeding as in case (i), find $F(z)$ by using the formula

$$F(z)=\int\left[U_x(z,0)-iU_y(z,0)\right]dz+C$$

(c) Find $f(z)$ by using $f(z)=\frac{F(z)}{1+i}$

Case(iv): If $u+v$ is given :

(a) we have $f(z)=u+iv$ .....(1)

$$if(z)=iu-v \qquad \text{.....(2)}$$

Adding (1) and (2), we have

$(1+i)f(z)=(u-v)+i(u+v)$ or

$F(z)=U+iV$, where $F(z)=(1+i)f(z)$, $U=u-v$, $V=u+v$

Now the problem reduces to construct an analytic function $F(z)$ whose imaginary part is given.

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(b) Proceeding as in case (ii), find $F(z)$ by using the formula

$$F(z)=\int\left[V_y(z,0)+iV_x(z,0)\right]dz+C$$

(c) Find $f(z)$ by using $f(z)=\dfrac{F(z)}{1+i}$

Exercise 2.2

1. Find the regular function whose imaginary part is $v=e^x(x\sin y+y\cos y)$.
2. Show that the function $u(x,\,y)=x^3-3xy^2$ is harmonic and find the corresponding analytic function.
3. Show that the function $u(x,\,y)=e^x\cos y$ is harmonic. Determine its harmonic conjugate $v(x,\,y)$ and the analytic function $f(z)=u+iv$.
4. Determine the analytic function whose real part is $x^3-3xy^2+3x^2-3y^2+2x+1$.
5. Show that $u(x,\,y)=e^{-x}(x\sin y-y\cos y)$ is harmonic and find $v(x,\,y)$ such that $f(z)=u+iv$ is analytic.
6. Prove that $\phi(x,\,y)=\log\left\{(x-1)^2+(y-2)^2\right\}$ is harmonic in every region which does not include the point $(1,\,2)$.
7. Show that $u=\dfrac{1}{2}\log(x^2+y^2)$ is harmonic and find its harmonic conjugate.
8. If $f(z)=u+iv$ is an analytic function of z, find $f(z)$ if $u-v=(x-y)(x^2+4xy+y^2)$
9. If $u+v=\dfrac{\sin 2x}{\cosh 2y-\cos 2x}$ and $f(z)=u+iv$ is an analytic function of z, then find $f(z)$ in terms of z.
10. If $w=\phi+i\psi$ represents the complex potential for an electric field and $\psi=x^2-y^2+\dfrac{x}{x^2+y^2}$, determine the function ϕ.
11. Let $u(x,\,y)=e^x(x\cos y-y\sin y)$ be the given functions. Show that it is possible to find a function $v(x,\,y)$ such that $f(z)=u(x,\,y)+iv(x,\,y)$ is analytic, and then find $v(x,\,y)$ and $f(z)$ by both the methods discussed above. Finally note the essential difference between these two methods.
12. Prove that $u=y^3-3x^2y$ is a harmonic function. Determine its harmonic conjugate and find the

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corresponding analytic function $f(z)$ in terms of z.

13 . Find the analytic function whose real part is given and hence the imaginary part :

(i) $e^x \sin y$ (ii) $\sin x \cosh y$

14 . Find the analytic function whose imaginary part is given and hence find the real part.

(i) $\cos x \cosh y$ (ii) $\dfrac{x-y}{x^2+y^2}$

15. If $u+v = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$, find the corresponding analytic function $f(z)=u+iv$.

16. If $f(z)=u+iv$ is an analytic function of $z=x+iy$ and $u-v=\dfrac{e^y-\cos x+\sin x}{\cosh y-\cos x}$, find $f(z)$ subject to the condition $f\left(\dfrac{\pi}{2}\right)=\dfrac{3-i}{2}$.

Answers

1. $f(z)=ze^z+c$ 2. $f(z)=z^3+c$ 3. $v(x,\,y)=e^x\sin y+c,\ f(x)=e^z+d$ where $d=ic$

4. $f(z)=z^3+3z^2+2z+c$ 5. $v(x,\,y)=e^{-x}(x\cos y+y\sin y)+c'$ where $c'=\dfrac{c}{i}$

7. $v=\tan^{-1}\left(\dfrac{y}{x}\right)+c$ 8. $f(z)=-iz^3+c$ 9. $f(z)=\dfrac{\cot z}{1-i}+c'$, where $c'=\dfrac{c}{1+i}$

10. $\phi=-2xy+\dfrac{y}{x^2+y^2}+c$ 11. $v(x,\,y)=e^x(x\sin y+y\cos y)+c$

12. $f(z)=i(z^3+c)$ 13. (i) $v(x,\,y)=-e^x\cos y+\dfrac{c}{i}$ (ii) $v(x,\,y)=\cos x\sinh y+\dfrac{c}{i}$

14. (i) $\sin x\sinh y+c$ (ii) $\dfrac{x+y}{x^2+y^2}+c$

15. $f(z)=\dfrac{i}{1+i}\cot z+\dfrac{c}{1+i}=\dfrac{1}{2}(1+i)\cot z+c'$, where $c'=\dfrac{c}{1+i}$ 16. $f(z)=\cot\dfrac{1}{2}z+\dfrac{1}{2}(1-i)$

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2.3 Some more topics

Analyticity of $f(z), f(\overline{z}), \overline{f(z)}, \overline{f(\overline{z})}$:

1. If $f(z)$ is analytic then $f(\overline{z})$ is analytic iff f is a constant function.
2. If $f(z)$ is analytic then $\overline{f(z)}$ is analytic iff f is a constant function.
3. $\overline{f(\overline{z})}$ is analytic iff $f(z)$ is analytic.
4. $\overline{f(z)}$ is analytic iff $f(\overline{z})$ is analytic.
5. If $f(z)$ is analytic in D then $f(z)$ is bounded on every bounded subset of D.

Results : Suppose $f = u + iv$ be analytic then

1. If $u =$ constant, then f is constant.
2. If $v =$ constant, then f is constant.
3. If f is real valued then f is constant.
4. If f is imaginary valued then f is constant.
5. If $\overline{f}$ is analytic then f is constant.
6. If $|f|$ is constant then f is a constant function.
7. If $|f|$ is analytic then f is a constant function.
8. If Image of f lies on straight line then f is constant.
9. If Image of f lies on until circle then f is constant.
10. If $\text{Arg}(f)$ is constant then f is constant.
11. If $f'(z)$ is zero then f is constant.
12. If u and v both are the function of only x (only y) then f is constant.
13. If u and v are harmonic conjugate of each other then both are constant functions. Hence f is constant.
14. If $u^2 = v$ $\left(\text{or } v^2 = u\right)$ then f is constant.
15. If $f(z)$ is entire and real part of $f(z)$ is bounded then $f(z)$ is constant.
16. Let $f(z)$ is entire and $\exists$ non-zero z_1 and $z_2 \in \mathbb{C}$ such that $f(z + z_1) = f(z + z_2) = f(z)\ \forall\ z \in \mathbb{C}$ and $\frac{z_1}{z_2}$ is not real then $f(z)$ is constant.

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True-false exercise

1. If f is a complex function for which $\lim_{z\to 2+i} \mathrm{Re}(f(z)) = 4$ and $\lim_{z\to 2+i} \mathrm{Im}(f(z)) = -1$, then $\lim_{z\to 2+i} f(z) = 4 - i$.

2. If f is a complex function for which $\lim_{x\to 0} f(x+0i) = 0$ and $\lim_{y\to 0} f(0+iy) = 0$, then $\lim_{z\to 0} f(z) = 0$.

3. If f is a complex function that is continuous at the point $z = 1+i$, then the function $g(z) = 3[f(z)]^2 - (2+i)f(z) + i$ is continuous at $z = 1+i$.

4. If f is a complex function that is continuous on the entire complex plane, then the function $g(z) = \overline{f(z)}$ is continuous on the entire complex plane.

5. If a complex function f is differentiable at point z, then f is analytic at z.

6. The function $f(z) = \dfrac{y}{x^2+y^2} + i\dfrac{x}{x^2+y^2}$ is differentiable for all $z \neq 0$.

7. The function $f(z) = z^2 + \overline{z}$ is nowhere analytic.

8. The function $f(z) = \cos y - i \sin y$ is nowhere differentiable.

9. There does not exist an analytic function $f(z) = u(x,y) + iv(x,y)$ for which $u(x,y) = y^3 + 5x$.

10. The function $u(x,y) = e^{4x}\cos 2y$ is the real part of an analytic function.

11. If $f(z) = e^x \cos y + ie^x \sin y$, then $f'(z) = f(z)$.

12. If $u(x,y)$ and $v(x,y)$ are harmonic function in a domain D, then the function $f(z) = \left(\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}\right) + i\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right)$ is analytic in D.

13. If g is an entire function, then $f(z) = \left(iz^2 + z\right)g(z)$ is necessarily an entire function.

14. The Cauchy-Riemann equations are the necessary conditions for differentiability.

15. The Cauchy-Riemann equations can be satisfied at a point z, but the function $f(z) = u(x,y) + iv(x,y)$ can be non-differentiable at z.

16. If the function $f(z) = u(x,y) + iv(x,y)$ is analytic at a point z, then necessarily the function $g(z) = v(x,y) - iu(x,y)$ is analytic at z.

17. A real-valued function $u(x,y)$ is harmonic in D iff $u(x,-y)$ is harmonic in D.

18. The function $\sin \overline{z}$ is nowhere analytic.

19. If f is analytic at z_0, then f''' necessarily exists at z_0.

20. The function $f(x+iy) = x^3 + ax^2y + bxy^2 + cy^3$ is analytic in $\mathbb{C}$ only if $a = 3i$, $b = -3$ and $c = -i$.

21. The functions $f(x+iy) = x^2 - y^2 + x + i(2xy + y)$ and $g(x+iy) = x^2 + y^2 + x + i(2xy + y)$ represent analytic functions in $\mathbb{C}$.

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22. If f and g are functions that satisfy the C-R equations at a point $a \in \mathbb{C}$, then $f+g$ and fg also satisfy the C-R equations at a.

23. The function f defined by
$$f(z) = u + iv = \begin{cases} \dfrac{\mathrm{Im}(z^2)}{\overline{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$
satisfies the C-R equations at the origin, yet it is not differentiable there.

24. The function f defined by
$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ \exp(-1/z^4) & \text{if } z \neq 0 \end{cases}$$
is continuous and satisfies the C-R equations at the origin.

25. The function $f(z)$ is analytic in a domain D iff both real and imaginary parts of $f(z)$ and $zf(z)$ are harmonic in D.

26. The function $f(x+iy) = x^3 + i(y-1)^3$ is nowhere analytic but is continuous in $\mathbb{C}$ and differentiable on $\{x+iy : x+y=1\} \cup \{x+iy : x-y=-1\}$.

27. An entire function f such that $f(x+iy) = u(x) + iv(y)$ must be of the form $f(z) = \alpha z + \beta$ with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

28. There exists an analytic functions f such that $\mathrm{Re}\, f(z) = y^2 - 2x$.

29. There exists an analytic functions f such that $\mathrm{Im}\, f(z) = x^3 - y^3$.

Assignment

---------- S C Q ----------

1. If the function $f(z)$ is continuous at z_0, then
   1. $f(z)$ is differentiable at z_0
   2. $f(z)$ is not necessarily differentiable at z_0
   3. $f(z)$ is analytic at z_0
   4. None of the above

2. If a function $f(z)$ is analytic at a point $z = z_0$, then which of the following statements is false?
   1. f is differentiable at z_0
   2. f is not continuous at z_0
   3. f is defined at z_0
   4. f is continuous at z_0

3. The function $f(z) = |z|^2$ is
   1. everywhere analytic
   2. nowhere analytic
   3. analytic at $z = 0$
   4. None of these

4. The function $f(z) = |z|^2$ is differentiable at
   1. $z = 0$
   2. $z \neq 0$
   3. nowhere
   4. None of these

5. $f(z) = e^y(\cos x + i \sin x)$ is
   1. an entire function
   2. analytic in $x^2 + 4y^2 < 24$
   3. nowhere analytic
   4. differentiable everywhere except $z = 0$

6. If $f(z)$ is analytic in a domain D, then
   1. $f^{(n)}z$ exist in D
   2. $f^{(n)}z$ does not exist in D
   3. $f^{(n)}z = 0$ for all n in D
   4. None of the above

7. The only function among the following, that is analytic, is

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1. $f(z)=\mathrm{Re}(z)$ 2. $f(z)=\mathrm{Im}(z)$
3. $f(z)=\bar{z}$ 4. $f(z)=\sin z$

8. The Cauchy-Riemann equations for $f(z)=u(x,y)+iv(x,y)$ to be analytic are
1. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$
2. $\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$
3. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$
4. $\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$

9. Polar form of C-R equations are
1. $\frac{\partial u}{\partial \theta}=\frac{1}{r}\frac{\partial v}{\partial r}, \frac{\partial u}{\partial r}=r\frac{\partial v}{\partial \theta}$
2. $\frac{\partial u}{\partial \theta}=r\frac{\partial v}{\partial r}, \frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$
3. $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$
4. $\frac{\partial u}{\partial r}=r\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$

10. If $w=u(x,y)+iv(x,y)$ is an analytic function of $z=x+iy$, then $\frac{dw}{dz}$ equals
1. $i\frac{\partial w}{\partial x}$ 2. $-i\frac{\partial w}{\partial x}$
3. $i\frac{\partial w}{\partial y}$ 4. $-i\frac{\partial w}{\partial y}$

11. In order that the function $f(z)=\frac{|z|^2}{z}, z\neq 0$ be continuous at $z=0$, we should define $f(0)$ equal to
1. 2 2. –1
3. 0 4. 1

12. If $f(z)$ is analytic and equal $u(x,y)+iv(x,y)$, then $f'(z)$ equals
1. $\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$ 2. $\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}$
3. $\frac{\partial u}{\partial y}-i\frac{\partial v}{\partial x}$ 4. None of these

13. The value of *m* so that $2x-x^2+my^2$ may be harmonic is
1. 0 2. 1
3. 2 4. 3

14. A function is said to be harmonic, if
1. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}=0$ 2. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$
3. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$ 4. $\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial y}=0$

15. *u, v* are called conjugate harmonic functions, if
1. *u, v* are harmonic functions and $u+iv$ is analytic function
2. u,v are harmonic functions and $u+iv$ may not be analytic function.
3. u,v are harmonic functions.
4. $u+iv$ is analytic function.

16. If $u+iv$ is analytic function, then dv is equal to
1. $\frac{\partial v}{\partial x}dx-\frac{\partial v}{\partial y}dy$ 2. $-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$
3. $\frac{\partial u}{\partial x}dx-\frac{\partial u}{\partial y}dy$ 4. $\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$

17. An analytic function is
1. infinitely differentiable
2. finitely differentiable
3. not differentiable
4. None of the above

18. Let f be an analytic in a region and at every point either $f=0$ or $f'=0$ then

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1. f is constant
2. f is polynomial
3. f may be non-constant
4. None of above

19. If $f(z)$ is entire function such that

$f(z+1)=f(z+i)=f(z)\ \forall\ z\in\mathbb{C}$ and

$f(3+5i)=2+6i$ the $f\left(\frac{1}{2}+\frac{1}{2}i\right)=?$

1. $2+6i$
2. $1+3i$
3. 0
4. None of these

20. The function $f:\mathbb{C}\to\mathbb{C}$ defined by

$f(z)=e^{\bar{z}}$ is

1. an entire function
2. analytic in unit disc $\{z:1\le 1<1\}$
3. not analytic at any point
4. none of the above

21. Let $f:\mathbb{C}\to\mathbb{C}$ be a complex valued function given by $f(z)=u(x,y)+iv(x,y)$. Suppose that $v(x,y)=3xy^2$. Then
1. f cannot be holomorphic on $\mathbb{C}$ for a suitable choice of u.
2. f is holomorphic on $\mathbb{C}$ for a suitable choice of u.
3. f is holomorphic on $\mathbb{C}$ for all choices of u.
4. v is not differentiable as a function of x and y.

(CSIR NET June 2011)

22. Let $f:\mathbb{C}\to\mathbb{C}$ be a complex valued function of the form $f(x,y)=u(x,y)+iv(x,y)$. Suppose that $u(x,y)=3x^2y$. Then
1. f cannot be holomorphic on $\mathbb{C}$ for any choice of v.
2. f is holomorphic on $\mathbb{C}$ for a suitable choice of v.
3. f is holomorphic on $\mathbb{C}$ for all choices of v
4. u is not differentiable.

(CSIR NET June 2012)

23. Let f be a real valued harmonic function on $\mathbb{C}$, that is, f satisfies the equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$. Define the functions

$g=\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}$ $\qquad$ $h=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}$

Then
1. g and h are both holomorphic functions/
2. g is holomorphic, but h need not be holomorphic.
3. h is holomorphic, but g need not be holomorphic.
4. both g and h are identically equal to the zero function.

(CSIR NET June 2015)

24. Let $u(x,y)=2x(1-y)$ for all real x and y. Then, a function $v(x,y)$, so that $f(z)=u(x,y)+iv(x,y)$ is analytic, is

1. $x^2-(y-1)^2$
2. $(x-1)^2-y^2$
3. $(x-1)^2+y^2$
4. $x^2+(y-1)^2$

(GATE 2010)

25. Consider the functions $f(z)=x^2+iy^2$ and $g(z)=x^2+y^2+ixy$. At $z=0$,
1. f is analytic but not g
2. g is analytic but not f
3. both f and g are analytic
4. neither f nor g is analytic

(GATE 2005)

26. Consider a function $f(z)=u+iv$ defined on $|z-i|<1$, where u,v are real valued functions of x,y. Then, $f(z)$ is analytic for u equals to

1. x^2+y^2
2. $\ln(x^2+y^2)$
3. e^{xy}
4. $e^{x^2-y^2}$

(GATE 2003)

27. At $z=0$, the function $f(z)=z^2\bar{z}$
1. does not satisfy Cauchy Riemann equations
2. satisfies Cauchy Riemann equations but is not differentiable
3. is differentiable

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4. is analytic

(GATE 2003)

28. Which of the following is not the real part of an analytic function ?

1. $x^2 - y^2$
2. $\dfrac{1}{1+x^2+y^2}$
3. $\cos x \cosh y$
4. $x + \dfrac{x}{x^2+y^2}$

(GATE 2006)

29. The function $\sin z$ is analytic in

1. $\mathbb{C} \cup \{\infty\}$
2. $\mathbb{C}$ except on the negative real axis
3. $\mathbb{C} - \{0\}$
4. $\mathbb{C}$

(GATE 2001)

---------- **M C Q** ----------

1. Let $f(z) = |\sin z|$ be a complex function.

Let S denote the set of regular (analytic) points of f then

1. $S = \phi$
2. S is countable
3. S is uncountable
4. $S = \mathbb{C}$

2. Let $f(z) = u(x,y) + iv(x,y)$ be a complex analytic function. Let $f_1(z) = u(x,y)$, $f_2(z) = v(x,y)$, $f_3(z) = u(x_1 - y) + iv(x_1 - y)$ then which of following is a sufficient condition for f to be a constant function.

1. $f_1(z)$ is analytic
2. $f_2(z)$ is analytic
3. $f_3(z)$ is analytic
4. All of above

3. The function $f(z) = \sec z$ is

1. analytic for all z
2. analytic for $z = \pi$
3. not analytic at $z = \dfrac{\pi}{2}$
4. None of these

4. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. For $z = x + iy$, let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be such that $u(x,y) = \text{Re } f(z)$ and $v(x,y) = \text{Im } f(z)$. Which of the following are correct ?

1. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$
2. $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$
3. $\dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial^2 u}{\partial y \partial x} = 0$
4. $\dfrac{\partial^2 v}{\partial x \partial y} + \dfrac{\partial^2 v}{\partial y \partial x} = 0$

(CSIR NET June 2013)

5. Let $u(x+iy) = x^3 - 3xy^2 + 2x$. For which of the following functions v, is $u + iv$ a holomorphic function on $\mathbb{C}$?

1. $v(x+iy) = y^3 - 3x^2 y + 2y$
2. $v(x+iy) = 3x^2 y - y^3 + 2y$
3. $v(x+iy) = x^3 - 3xy^2 + 2x$
4. $v(x+iy) = 0$

(CSIR NET Dec 2014)

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True false key

1. T	2. T	3. T	4. T
5. F	6. T	7. T	8. T
9. T	10. F	11. T	12. T
13. T	14. T	15. T	16. T
17. T	18. T	19. T	20. T
21. F	22. T	23. T	24. F
25. T	26. T	27. T	28. F
29. F			

Assignment key

S C Q

1. 2	2. 2	3. 2	4. 1
5. 3	6. 1	7. 4	8. 3
9. 3	10. 4	11. 3	12. 1
13. 2	14. 2	15. 1	16. 2
17. 1	18. 1	19. 1	20. 3
21. 1	22. 1	23. 2	24. 1
25. 4	26. 2	27. 3	28. 2
29. 4			

M C Q

1. 1,2	2. 1,2,3,4	3. 2,3	4. 1,2,3
5. 2			

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Chapter - 3

Complex Integration

3.1 Complex Integration by Direct Computation

Arc : The equation $z(t) = x(t) + i\,y(t)$, $a \le t \le b$ where $x(t)$ and $y(t)$ are continuous function of t, represents an arc in the complex plane. If the arc is continuous and differentiable then it is called regular arc.

Simple Arc: An arc $z(t) = x(t) + i\,y(t)$ is said to be a simple arc if $z(t_1) = z(t_2)$ only when $t_1 = t_2$.

Closed contour: A curve which is composed of a finite number of regular arcs is called a contour. By a closed contour, we mean a closed simple arc consisting of a finite number of regular arcs i.e., the contour is closed and does not intersect itself.

Rectifiable arc : Let $z = x(t) + i\,y(t)$ be the equation of a given arc, the range of t being $t_0 \le t \le T$.

Let z_0 , z_1 , z_2 , ... , z_n be the points of this arc corresponding to the values t_0 , t_1 , t_2 , ... , t_n of t where $t_0 < t_1 < t_2 < ..... < t_n = T$. Evidently the length of the polynomial arc defined by joining z_0 and z_1 , z_1 and z_2 by straight line segments is given by $\sum_{r=1}^{n} |z_r - z_{r-1}|$.

If this sum tends to a unique finite limit 'l' as $n \to \infty$, then we say that the arc L defined by $z = x(t) + i\,y(t)$ is rectifiable and its length is 'l'.

Results : If L is a rectifiable arc joining the points $z = a$ and $z = b$, then

1. $\int_L dz = b - a$
2. $\int_L |dz| =$ length of arc L
3. $\int_L z\,dz = \frac{b^2 - a^2}{2}$

Common parameterizations in complex plane :

Line : A parameterization of the line containing the points z_0 and z_1 is: $z(t) = z_0(1-t) + z_1 t$; $-\infty < t < \infty$.

Line Segment : A parameterization of the line segment from z_0 to z_1 is: $z(t) = z_0(1-t) + z_1 t$; $0 \le t \le 1$.

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Ray : A parameterization of the ray emanating from z_0 and containing z_1 is: $z(t) = z_0(1-t) + z_1 t$;

$0 \le t < \infty$

Circle : A parameterization of the circle centered at z_0 with radius r is : $z(t) = z_0 + r(\cos t + i \sin t)$;

$0 \le t \le 2\pi$

Exercise 3.1

1. Evaluate the following :

(i) $\int_C (z+3)\,dz$; $C : x = 2t$, $y = 4t - 1$, $1 \le t \le 3$

(ii) $\int_C (2\bar{z} - z)\,dz$; where C is $x = -t$, $y = t^2 + 2$, $0 \le t \le 2$

(iii) $\int_C z^2\,dz$; where C is $z(t) = 3t + 2it$, $-2 \le t \le 2$

(iv) $\int_C |z|^2\,dz$, C: $x = t^2$, $y = \frac{1}{t}$, $1 \le t \le 2$

(v) $\int_C \mathrm{Re}\, z\,dz$, C: $|z| = 1$

2. Evaluate the following integrals :

(i) $\int_C (x^2 + i y^3)\,dz$, where C is the straight line from $z = 1$ to $z = i$

(ii) $\int_C e^z\,dz$; where C is the polygonal path consisting of line segments from $z = 0$ to $z = 2$ and $z = 2$ to $z = 1 + \pi i$.

(iii) $\int_C (z^2 - z + 2)\,dz$ where C is the line segment from i to 1.

3. Evaluate the following integrals :

(i) $\int_C \frac{z+1}{z}\,dz$, where C is the right half of the circle $|z| = 1$ from $z = -i$ to $z = i$

(ii) $\int_C \left(\frac{1}{(z+i)^3} - \frac{5}{z+i} + 8 \right) dz$, where C is the circle $|z+i| = 1$

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(iii) $\int_C \frac{1}{z-z_0}dz$ where C is the circle $|z-z_0|=r$

(iv) $\int_C \frac{1}{(z-z_0)^2}dz$ where C is the circle $|z-z_0|=r$

Answers

1. (i) $-28+84i$ (ii) $50+\frac{44i}{3}$ (iii) $\frac{16}{3}(3+2i)^3$

   (iv) $21+2\log 2-\frac{21i}{8}$ (v) πi

2. (i) $-\frac{7}{12}+\frac{i}{12}$ (ii) $-e-1$ (iii) $\frac{4}{3}-\frac{5}{3}i$

3. (i) $i(\pi+2)$ (ii) $-10\pi i$ (iii) $2\pi i$ (iv) 0

3.2 Cauchy's Theorem and Cauchy's Integral Formula

Result 1. Cauchy's Theorem : Suppose f is a function which is analytic in a simple connected domain D. Then for every simple closed contour C in D $\int_C f(z)dz=0$.

Result 2. Deformation of contour : Suppose $C, C_1, C_2, ..., C_n$ are closed curves with a positive orientation such that $C_1, C_2, ..., C_n$ are interior to C but the regions interior to each C_k, $k=1,2,...,n$ have no point in common. If f is analytic on each contour and each point interior to C but exterior to all the C_k, $k=1,2,...,n$ then $\int_C f(z)dz=\sum_{k=1}^{n}\int_{C_k} f(z)dz$

Result 3. $\frac{1}{(z-z_0)^n}$ result :

(i) If C is closed contour and z_0 is an external point of C then $\int_C \frac{1}{(z-z_0)^n}dz=0$

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(ii) If C is closed contour and z_0 is interior point of C then $\int_C \frac{1}{(z-z_0)^n} dz = \begin{cases} 2\pi i, & \text{if } n=1 \\ 0, & \text{if } n \neq 1 \end{cases}$

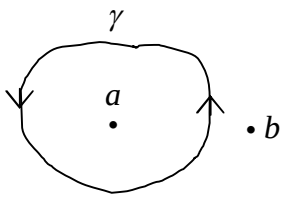
Result 4. Cauchy's Integral Formula : Suppose f is an analytic function in simply connected domain D and C is any simple closed contour lying entirely with in D. Then for any point z_0 with in C,

$$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i \; f(z_0).$$

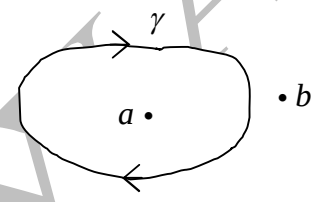
Result 5. Cauchy's Integral Formula for higher order derivatives : If f is an analytic function in a simple connected domain D, and C is simply closed contour lying entirely with in D. Then for any point z_0 with in C, we have $\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} f(z_0)$

Def. Winding number : Suppose that γ is a closed contour in $\mathbb{C}$ and a be a given point in $\mathbb{C} - \gamma$ i.e., a does not lie on γ. Then the winding number of γ about the point a is defined to be the number of times γ wraps around a in the anti-clockwise direction and it is denoted by $n(\gamma, a)$.

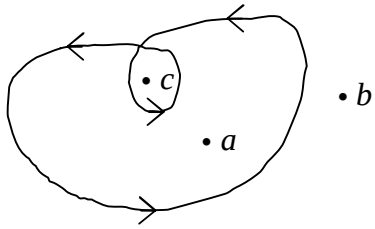
For illustration, consider the following :



$n(\gamma, a) = 1, \quad n(\gamma, b) = 0$



$n(\gamma, a) = -1, \quad n(\gamma, b) = 0$



$n(\gamma, a) = 1, \quad n(\gamma, b) = 0, \quad n(\gamma, c) = 2$

Result 6 : Cauchy integral formula in terms of winding number : Let f be analytic function within and on a closed contour γ and a is an interior point of γ, then $\int_\gamma \frac{f(z)}{z-a} dz = 2\pi i \; n(\gamma, a) f(a)$

Result 7 : Morera's Theorem : If $f(z)$ is continuous in a region D and if the integral $\int_C f(z)\,dz$ taken around any closed contour in D vanishes, then $f(z)$ is analytic in D.

Remark : Morera theorem is known as short converse or modified converse or refined converse of Cauchy theorem.

Result 8 : Cauchy inequality : If $f(z)$ is analytic on and within a circle C given by $|z - a| = R$ and if the function is bounded by M i.e. $|f(z)| \leq M$ for all $z \in C$, then $\left| f^{(n)}(a) \right| \leq \frac{M.n!}{R^n}$

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Result 9. Liouville's theorem : An entire and bounded function is always constant.

OR

A non constant entire function is always unbounded.

Result 10. The functions e^z, $\sin z$, $\cos z$, $\sinh z$ and $\cosh z$ are unbounded.

Result 11. If an entire function is bounded on a half-plane then it need not be constant.

e.g., consider the function $f(z)=e^z$ on the set $S=\{z:\mathrm{Re}(z)\le 0\}$. Similarly, take $g(z)=e^{iz}$ on $S=\{z:\mathrm{Im}(z)\ge 0\}$.

Result 12. If $f:\mathbb{C}\to D$ is an analytic function where D is the open unit disc then f is constant.

Result 13. M-L Inequality : If $f(z)$ is continuous on a contour C such that $|f(z)|\le M$ for all z on C, then $\left|\int_C f(z)\,dz\right|\le ML$, where L is the length of the contour C.

Exercise 3.2

Evaluate the following :

1. $\int_C \left(z^3-1+3i\right)dz$; C is $|z|=1$

2. $\int_C \left(z^2+\frac{1}{z-4}\right)dz$; C is $|z|=1$

3. $\int_C \left(\frac{z}{2z+3}\right)dz$; $C:|z|=1$

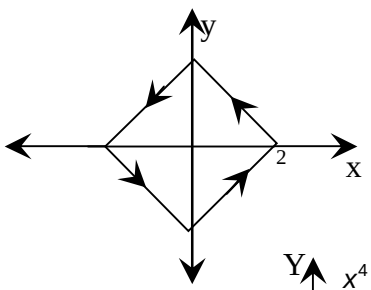
4. $\int_C \left(\frac{z-3}{z^2+2z+2}\right)dz$; $C:|z|=1$

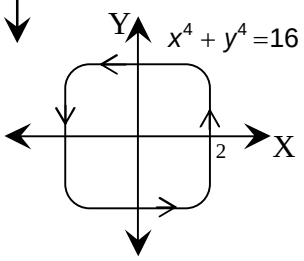
5. $\int_C \frac{\sin z}{\left(z^2-25\right)\left(z^2+9\right)}dz$; $C:|z|=1$

6. $\int_C \tan z\,dz$; $C:|z|=1$

7. $\int_C \left(\frac{z^2+9}{\cosh z}\right)dz$; $C:|z|=1$

8. $\int_C \frac{1}{z}dz$; where C is given in the figure





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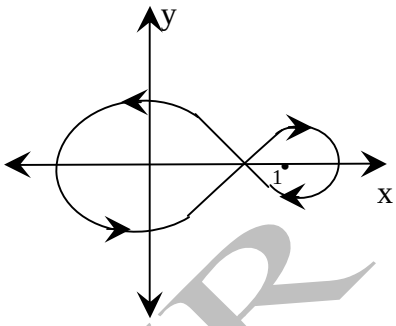
9. $\int_C \frac{5}{z+1+i}dz$; where C is the contour shown in the figure

10. $\int_C \frac{8z-3}{z^2-z}dz$; where C is the closed contour as given in figure:



11. $\int_C \left(z+\frac{1}{z}\right)dz$; $|z|=2$

12. $\int_C \left(z+\frac{1}{z^2}\right)dz$; $|z|=2$

13. $\int_C \frac{z}{z^2-\pi^2}dz$; $|z|=3$

14. $\int_C \frac{10}{(z+i)^4}dz$; $|z+i|=1$

15. $\int_C \frac{4}{z-3i}dz$; $|z|=5$

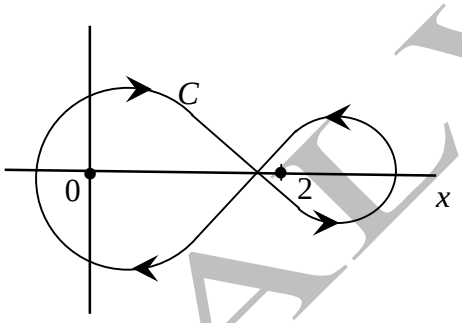
16. $\int_C \frac{e^z}{z-\pi i}dz$; $|z|=4$

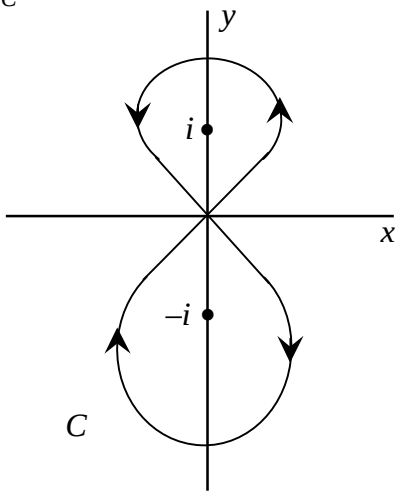
17. $\int_C \frac{1+e^z}{z}dz$; $|z|=1$

18. $\int_C \frac{z^2-3z+4i}{z+2i}dz$; $|z|=3$

19. $\int_C \frac{\cos z}{3z-\pi}dz$; $|z|=1.1$

20. $\int_C \frac{z^2}{z^2+4}dz$; (a) $|z-i|=2$ (b) $|z+2i|=1$





21. $\int_C \frac{3z+1}{z(z-2)^2}dz$ (above figure)

22. $\int_C \frac{e^{iz}}{(z^2+1)^2}dz$ (above figure)

23. $\int_C \frac{1}{z^2(z^2+1)}dz$; $|z-i|=\frac{3}{2}$

24. $\int_C \frac{1}{z^3(z-1)^2}dz$; $|z-2|=5$

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25. $\int_C \left(\frac{\cosh z}{(z-\pi)^3} - \frac{\sin^2 z}{(2z-\pi)^3} \right) dz \; ; \; |z| = 3$

26. $\int_C \left(\frac{e^{2iz}}{z^4} - \frac{z^4}{(z-i)^3} \right) dz \; ; \; |z| = 6$

27. $\int_C \frac{1}{z^3(z-4)} dz \; ; \; |z| = 1$

28. $\int_C \frac{z+2}{z^2(z-1-i)} dz \; ; \; |z| = 1$

29. $\int_C \frac{\cos 2z}{z^5} dz \; ; \; |z| = 1$

30. $\int_{|z|=1} \frac{2\,\mathrm{Re}(z)}{z+2} dz$

Answers

1. 0 2. 0 3. 0 4. 0 5. 0 6. 0 7. 0
8. $2\pi i$ 9. $10\pi i$ 10. $-4\pi i$ 11. $2\pi i$ 12. 0 13. 0 14. 0
15. $8\pi i$ 16. $-2\pi i$ 17. $4\pi i$ 18. $-20\pi - 8\pi i$ 19. $\frac{\pi}{3} i$
20. (a) -2π (b) 2π 21. $-\pi i$ 22. $\frac{\pi}{e}$ 23. $-\pi$ 24. 0 25. $\frac{\pi}{4} i$
26. $\frac{8\pi}{3} + 12\pi i$ 27. $-\frac{\pi}{32} i$ 28. $-3\pi - \pi i$ 29. $\frac{4}{3}\pi i$ 30. πi

---

3.3 Power series

Def. Power Series : An infinite series of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$ is called a power series. The co-efficients a_n's are complex constants and the point z_0 is called the centre of the power series.

If we substitute $z - z_0 = \xi$ then the above power series takes the form $\sum_{n=0}^{\infty} a_n \, \xi^n$.

Def. Absolute Convergence : The power series $\sum a_n \, z^n$ is said to be absolutely convergent if the series $\sum |a_n| \, |z|^n$ is convergent.

Def. Conditionally Convergent : The power series $\sum_{n=0}^{\infty} a_n \, \xi^n$ is said to be conditionally convergent if

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$\sum a_n z^n$ is convergent but $\sum |a_n||z|^n$ is not convergent.

Def. Radius of convergence : Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is a power series. A non-negative number R is said to be radius of convergence of given power series if the power series converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$.

Def. Circle of convergence : The circle $|z-z_0| = R$ in the above definition is called circle of convergence.

Formula : $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, provided the limit on the R.H.S. exist.

Result : Circle of convergence of a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is the largest circle centred at z_0 such that the series is convergent at every point inside that circle.

Result : Circle of convergence of a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is a circle centred at z_0 such that the series is convergent at every point inside the circle and divergent at every point outside the circle.

Cauchy Root Test : Let $\sum_{n=0}^{\infty} u_n$ be a series of complex numbers and $\lim_{n\to\infty} |u_n|^{\frac{1}{n}} = l$. Then the series is

(i) convergent if $l < 1$. (ii) divergent if $l > 1$. (iii) test fails at l=1.

D' Ratio Test : Let $\sum_{n=0}^{\infty} u_n$ be a series of complex numbers and $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = l$.

Then the series is

(i) convergent if $l < 1$. (ii) divergent if $l > 1$. (iii) test fails at l=1.

Investigation on the boundary :

1. Direct computation on some simple points.
2. Using Contra-positive Test : If $\lim_{n\to\infty} u_n \neq 0$ then the series is not convergent.
3. Geometric series never converges on the boundary.
4. Dirichlet Test: If the series $\sum_{n=1}^{\infty} a_n b_n$ is such that

   (i) $\left|\sum_{n=1}^{k} a_n\right| \leq M$ for all k *i.e.* partial sums of $\sum a_n$ are bounded.

   (ii) $\lim_{n\to\infty} b_n = 0$

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(iii) $\sum_{n=1}^{\infty}(b_n - b_{n+1})$ is convergent; then the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Some results on power series:

1. If the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some point $z = z_0$ then the power series converges for every point $z = z_1$ such that $|z_1| < |z_0|$.
2. If above power series diverges for $z = z_0$ then it diverges for every point $z = z_1$ such that $|z_1| > |z_0|$.
3. The sum function of a power series is analytic with in its circle of convergence and sum function has atleast one singularity on the circle of convergence.
4. A power series can be differentiated term by term with in its circle of convergence.
5. The radius of convergence of the differentiated power series is same as that of original series.
6. A power series can be integrated term by term with in its circle of convergence.
7. If radius of convergence of a power series $\sum a_n z^n$ is R then

   (i) Radius of convergence of $\sum {a_n}^k z^n$ is R^k. (ii) Radius of convergence of $\sum a_n z^{nk}$ is $R^{\frac{1}{k}}$.

   (iii) Radius of convergence of $\sum n^k a_n z^n$ is R. (iv) Radius of convergence of $\sum n^n a_n z^n$ is 0.
8. Radius of convergence of a power series is the distance of the nearest singularity of its sum function from the centre of the power series.
9. If $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ are two power series with radius of convergence R_1 and R_2 then the radius of convergence of the power series $\sum_{n=0}^{\infty} (a_n \pm b_n)(z - z_0)^n$ is $\min\{R_1, R_2\}$.

Exercise 3.3

1. Find the radius of convergence of the following power series :

   (i) $\sum \frac{z^n}{n^n}$ (ii) $\sum \frac{2^{-n} z^n}{1 + in^2}$ (iii) $\sum \frac{(\lfloor n)^k}{\lfloor kn} z^n$

   (iv) $\sum \frac{\lfloor n}{n^n} z^n$ (v) $\sum_{n=0}^{\infty} (3 + ni)^n z^n$ (vi) $\sum \frac{1}{n^p} z^n$

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(vii) $\sum_{n=1}^{\infty}(\log n)^n z^n$ (viii) $\sum_{n=0}^{\infty}\left(\frac{n\sqrt{2}+i}{1+2in}\right)z^n$ (ix) $1+\frac{a.b}{1.c}z+\frac{a(a+1)\,b(b+1)}{1.2.c(c+1)}z^2+....$

(x) $\frac{z}{2}+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+....$ (xi) $\sum_{n=1}^{\infty}\frac{z^n}{n}$ (xii) $\sum_{n=2}^{\infty}\frac{z^n}{n(\log n)^2}$

(xiii) $\sum_{n=1}^{\infty}\lfloor n\, z^n$ (xiv) $\sum_{n=1}^{\infty}\frac{(z+2)^{n-1}}{(n+1)^3\,4^n}$ (xv) $\sum\left(1-\frac{1}{n}\right)^2 z^n$

(xvi) $\sum\frac{z^n}{2^n+1}$ (xvii) $\sum\frac{(in+2)z^n}{2^n}$ (xviii) $\sum_{n=2}^{\infty}\frac{z^n}{\log n}$

(xix) $\sum(-1)^n\left(z^n+z^{n+1}\right)$ (xx) $\sum_{n=1}^{\infty}a_n z^n, a_n =$ number of divisors of n^{50}

(xxi) $\sum_{n=0}^{\infty}p(n)z^n, p(x)$ be a polynomial of the real variable x of degree $k\geq 1$

(xxii) $\sum_{n=1}^{\infty}(n\log n)z^n$ (xxiii) $\sum_{n=1}^{\infty}\frac{e^{n^2}}{n}z^n$ (xxiv) $\sum_{n=1}^{\infty}\frac{n(n-1)}{2^n}z^{n-2}$

(xxv) $\sum\frac{(-1)^n(z-2i)^n}{n}$

2. Find the domain of convergence of the following series and if the domain of convergence is a disc then find the radius of convergence :

(i) $\sum_{n=1}^{\infty}\left(\frac{iz-1}{2+i}\right)^n$ (ii) $\sum_{n=0}^{\infty}\frac{1}{\left(z^2+1\right)^n}$ (iii) $\sum_{n=1}^{\infty}\left(\frac{2i}{z+i+1}\right)^n$

(iv) $\sum_{n=0}^{\infty}2^{-n}z^{2n}$ (v) $\sum_{n=0}^{\infty}3^{-n}(z-1)^{2n}$

3. Find the domain of convergence of the following series and if the domain of convergence is a disc then find the radius of convergence :

(i) $\sum_{n=1}^{\infty}\frac{1.3.5....(2n-1)}{\lfloor n}\left(\frac{1-z}{z}\right)^n$ (ii) $\sum_{n=0}^{\infty}\frac{z^{4n}}{1+4n}$ (iii) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n-1}}{\lfloor 2n-1}$

(iv) $\sum_{n=1}^{\infty}z^{\lfloor n}$ (v) $\sum_{n=1}^{\infty}z^{n^2}$

4. Investigate the behavior on the boundary of circle of convergence of the following series :

(i) $\sum_{n=1}^{\infty}\frac{1}{n}z^n$ (ii) $\sum_{n=1}^{\infty}\frac{z^{4n}}{1+4n}$ (iii) $\sum_{n=1}^{\infty}\frac{1}{n^2}z^n$ (iv) $\sum_{n=0}^{\infty}n.z^n$

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5. Without expanding the following function about the indicated point and find the radius of convergence.

(i) $f(z)=\dfrac{4+5z}{1+z^2}$, $z_0=2+5i$ (ii) $f(z)=\cot z$, $z_0=\pi i$ (iii) $f(z)=\dfrac{1}{z}$, $z_0=1+i$

Answers

1. (i) $R=\infty$ (ii) $R=2$ (iii) $R=k^k$ (iv) $R=e$ (v) $R=0$

(vi) $R=1$ (vii) $R=0$ (viii) $R=1$ (ix) $R=1$ (x) $R=\dfrac{3}{2}$

(xi) $R=1$ (xii) $R=1$ (xiii) $R=0$ (xiv) $R=4$ (xv) $R=1$

(xvi) $R=2$ (xvii) $R=2$ (xviii) $R=1$ (xix) $R=1$ (xx) $R=1$

(xxi) $R=1$ (xxii) $R=1$ (xxiii) $R=0$ (xxiv) $R=2$ (xxv) $R=1$

2. (i) d.o.c. is $|z+i|<\sqrt{5}$ and r.o.c. is $\sqrt{5}$ (ii) d.o.c. is $|z^2+1|>1$

(iii) d.o.c. is $|z+1+i|>2$ (iv) d.o.c. is $|z|<\sqrt{2}$ and r.o.c. is $\sqrt{2}$

(v) d.oc. is $|z-1|<\sqrt{3}$ and r.o.c. is $\sqrt{3}$

3. (i) d.o.c. is $\left|z-\dfrac{4}{3}\right|<\dfrac{2}{3}$ and r.o.c. is $\dfrac{2}{3}$ (ii) d.o.c. is $|z|<1$ and r.o.c. is 1

(iii) d.o.c. is $\mathbb{C}$ and r.o.c. is ∞ (iv) d.o.c. is $|z|<1$ and r.o.c. is 1

(v) d.o.c. is $|z|<1$ and r.o.c. is 1

4. (i) $|z|=1$, series is divergent at $z=1$ and convergent at all other points on the circle.

(ii) $|z|=1$, series is divergent at $z=1,-1,i,-i$ and convergent at all other points on the circle.

(iii) $|z|=1$ and series is convergent at all points of the circle.

(iv) $|z|=1$ and series is divergent at all points of the circle.

5. (i) $2\sqrt{5}$ (ii) π (iii) $\sqrt{2}$

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True-false exercise

1. The real line integral $\int_C (x^2+y^2)\,dx + 2xy\,dy$, where C is given by $y = x^3$ form (0, 0) to (1,1) has the same value on the curve $y = x^6$ from $(0,0)$ to $(1,1)$.

2. If f is analytic within and on a simple closed contour C, then $\oint_C f(z)\,dz = 0$.

3. If $\oint_C f(z)\,dz = 0$ for every simple closed contour C, then f is analytic within and on C.

4. The value of $\int_C \frac{z-2}{z}\,dz$ is the same for any path C in the right half-plane $\mathrm{Re}(z) > 0$ between $z = 1+i$ and $z = 10+8i$.

5. If g is entire, then $\oint_C \frac{g(z)}{z-i}\,dz = \oint_{C_1} \frac{g(z)}{z-i}\,dz$, where C is the circle $|z| = 3$ and C_1 is the ellipse $x^2 + \frac{1}{9}y^2 = 1$.

6. $\oint_C \frac{1}{(z-z_0)(z-z_1)}\,dz = 0$ for every simple closed contour C that encloses the points z_0 and z_1.

7. If f is analytic within and on the simple closed contour C and z_0 is a point within C, then $\oint_C \frac{f'(z)}{(z-z_0)}\,dz = \oint_C \frac{f(z)}{(z-z_0)^2}\,dz$.

8. $\oint_C \frac{1}{z^2+1}\,dz = 0$, where C is the ellipse $x^2 + \frac{1}{4}y^2 = 1$.

9. The function $f(z) = \cos z$ is entire and not a constant and so must be unbounded.

10. The power series $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges absolutely at every point on its circle of convergence.

11. A function f is analytic at a point z_0 if f can be expanded in a convergent power series centered at z_0.

12. The power series $\sum_{k=0}^{\infty} a_k z^k$ and $\sum_{k=1}^{\infty} k a_k z^{k-1}$ have the same radius of convergence R.

13. $\lim_{n\to\infty} (n!)^{1/n} = \infty$

14. If R is the radius of convergence of $\sum_{n\geq 0} a_n z^n$ then the radii of convergence of $\sum_{n\geq 0} a_n z^{2n}$ and $\sum_{n\geq 0} a_n^2 z^n$ are $\sqrt{R}$ and R^2, respectively.

15. The radius of convergence of the series $\sum_{n\geq 1} \frac{(-1)^n}{n(n+1)} z^{n(n+1)}$ is 1.

16. The series $\sum_{n=0}^{\infty} \left(\frac{z}{1-z}\right)^n$ converges for $\mathrm{Re}\, z \geq 1/2$.

17. The series $\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^4}{z^n}\right)$ converges for $|z| > 1$ and diverges everywhere else.

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18. $I = \int_{|z|=r} |z-r||dz| = 8r^2$.

19. If f is a complex-valued continuous function on $\mathbb{C}$, then
$I = \int_{|z|=1} \frac{f(z)-f(1/z)}{z} dz = 0$.

20. $p(z)$ is a polynomial of degree *n* in *z* with complex coefficients, then
$I = \int_{|z|=1} p(\bar{z})\, dz = 2\pi i p'(0)$.

21. If $\{a_n\}_{n\geq 0}$ is a sequence of real numbers such that $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} a_n (z+2)^n$, then the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ is 3.

Assignment

---------- S C Q ----------

1. The integral of $\oint_C (z-z_0)^m dz$, where z_0 is inside *C* and *C* is a closed contour, is equal to
   1. 0 for $m=-1$ 2. $2\pi i$ for $m=-1$
   3. 2 for $m=-1$ 4. π for $m=-1$

2. The value of $\oint_C \frac{z^3-6}{2z-i} dz\, C: |z-i|=1$ is
   1. $\frac{\pi}{8}$ 2. $6\pi i$
   3. $\left(\frac{\pi}{8}\right) - 6\pi i$ 4. $-6\pi i - \left(\frac{\pi}{8}\right)$

3. If $f(z)$ is continuous in a simple connected domain D and if $\oint_C f(z)dz = 0$ for any closed contour in D, then
   1. $f(z)$ is non analytic in D
   2. $f(z)$ is analytic in D
   3. $f(z)$ is constant
   4. None of these

4. If $f(z)$ is analytic in a simply connected domain D,then for every closed path C in D
   1. $\oint_C f(z)dz = 0$ 2. $\oint_C f(z)dz = 1$
   3. $\oint_C f(z)dz \neq 0$ 4. $\oint_C f(z)dz \neq 1$

5. If $\gamma : [0,1] \to \mathbb{C}$ is a closed recitifiable curve and $a \notin \{\gamma\}$, then $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is
   1. an integer 2. rational number
   3. real number 4. complex number

6. *C* is closed contour $|z|=r$ and $n \neq -1$, then $\int_C z^n dz$ is equal to
   1. $2\pi i$ 2. 2π
   3. I 4. 0

7. The value of $\int_C \frac{1}{z} dz$, where *C* is circle $z = e^{i\theta}$, $0 \leq \theta \leq \pi$ is
   1. πi 2. $-\pi i$
   3. $2\pi i$ 4. 0

8. The value of $\frac{1}{2\pi i}\int_{|z|=3} \frac{e^z}{z-2} dz$ is
   1. 0 2. 1
   3. e^2 4. e^3

9. The value of the integral $\oint_C \frac{e^z}{z-2} dz$ where $C: |z|=3$ is
   1. $2\pi i e^2$ 2. $2\pi i$
   3. e^2 4. None of these

10. The value of $\oint_C \frac{(3z^2+7z+1)}{z+1} dz$, where *C* is $|z| = \frac{1}{2}$ is
   1. $2\pi i$ 2. 0
   3. πi 4. $\frac{\pi i}{2}$

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11. For every path between the limits $\int_{-2}^{-2+i}(2+z)^2\,dz$ is equal to

1. $\frac{i}{3}$ 2. $\frac{i}{2}$
3. $\frac{-i}{3}$ 4. $\frac{i}{4}$

12. If C is circle $|z-a|=r$, then $\int_C \frac{dz}{z-a}$ is

1. $2\pi i$ 2. $-2\pi i$
3. πi 4. 0

13. If C is a circle $|z|=1$, then $\int_C \bar{z}\,dz$ is

1. πi 2. $2\pi i$
3. 0 4. None of these

14. The value of $\int_{|z|=2}\frac{e^{2z}}{(z+1)}dz$ is

1. $2\pi i e^{-2}$ 2. $\frac{8\pi i}{3}e^{-2}$
3. $\frac{2\pi i}{3}e^{-2}$ 4. 0

15. The value of $\frac{1}{2\pi i}\int_C \frac{e^z\,dz}{(z-2)'}$ if C is the circle $|z|=3$ is

1. e 2. e^2
3. e^3 4. None of these

16. The integral $\int_{|z|=2}\frac{\cos z}{z^3}dz$ equals

1. πi 2. $-\pi i$
3. $2\pi i$ 4. $-2\pi i$

17. Let Γ denote the boundary of the square whose sides lie along $x=\pm 1$ and $y=\pm 1$, where Γ is described in the positive sense. Then, the value of $\int_\Gamma \frac{z^2}{2z+3}dz$ is

1. $\frac{\pi i}{4}$ 2. $2\pi i$
3. 0 4. $-2\pi i$

18. The value of $\int_0^{2+i}(\bar{z})^2\,dz$ along the line $2y=x$ is

1. $\frac{5}{3}(2+i)$ 2. $\frac{5}{3}(2-i)$
3. $(2-i)$ 4. None of these

19. If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within or on a closed curve C, then $\int_C f(z)dz=$

1. ∞ 2. 1
3. 0 4. None of these

20. The integral given below equals $\int_C \frac{e^{3z}}{(z-\pi i)}dz$, where C is circle $|z-1|=6$ is

1. $-2\pi i$ 2. 0 3. πi 4. $2\pi i$

21. If $f(z)$ be an analytic function inside and on the boundary of a triangle, then $\oint_\Delta f(z)dz=$

1. ∞ 2. 1
3. 0 4. None of these

22. If C is the curve $y=x^3-3x^2+4x-1$, joining the two points $(1,1)$ and $(2,3)$. The value of $\int_C(12z^2-4iz)dz$ is

1. $-156+38i$ 2. $156-38i$
3. $38+156i$ 4. $38-156i$

23. $\int_C z^2\,dz$, where C is the circle with centre 0 and radius 2 equals

1. $4\pi^2$ 2. $2\pi i$ 3. 4 4. 0

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24. Let Γ be any circle enclosing the origin an oriented counter clockwise, then the value of the integral $\int_{\Gamma} \frac{\cos z}{z^2} dz$ is

1. $2\pi i$
2. 0
3. $-2\pi i$
4. undefined

25. If $f(z)=\frac{z^2+5z+6}{z-2}$ and the path of integration is a circle with originated centre and radius *r*, the Cauchy's theorem is applicable, whenever *r* equals

1. 1
2. 2
3. 3
4. 1 and 2

26. The value of $\frac{1}{2\pi i}\int_C \frac{e^z}{z-3} dz$; where $C:(r-4\cos\theta)(r-2\cos\theta)=0$ is

1. e^3
2. $2e^3$
3. $\frac{1}{2}e^3$
4. 0

27. For the positively oriented unit circle $\int_{|z|=1} \frac{2\mathrm{Re}(z)}{z+2} dz =$

1. 0
2. πi
3. $2\pi i$
4. $4\pi i$

28. Let Γ be the curve $\Gamma = 2+4\cos\theta$, $(0 \le \theta \le 2\pi)$. If $I_1 = \int_{\Gamma} \frac{dz}{z-1}$ and $I_2 = \int_{\Gamma} \frac{dz}{z-3}$. Then,

1. $I_1 = I_2$
2. $I_1 = 2I_2$
3. $2I_1 = I_2$
4. $I_1 = 0,\ 2I_2 \neq 0$

29. Putting $z = e^{i\theta}$, the integral $\int_0^{2\pi} \frac{\sin 3\theta}{5-3\cos\theta} d\theta$ reduces to…, where *C* is the circle $|z|=1$. Here… means

1. $\oint_C \frac{(z^3-1)dz}{z^3(z-3)}$
2. $\oint_C \frac{(z^6-1)dz}{z^3(3z-1)}$
3. $\oint_C \frac{(z^6-1)dz}{z^3(z-3)(3z-1)}$
4. None of these

30. The value of $\int_C \frac{e^{-z}}{z^2} dz$, where *C* is the circle $|z|=1$ is

1. -1
2. $2\pi i$
3. $-2\pi i$
4. None of these

31. The radius of convergence of the series given as $1+z+\frac{z^2}{2}-\frac{z^3}{3!}+\frac{z^4}{2^2}+\frac{z^5}{5!}+\frac{z^6}{2^3}-\frac{z^7}{7!}+\ldots$ is

1. 2
2. $\sqrt{2}$
3. ∞
4. 0

32. The radius of convergence of the power series of the function $f(z)=\frac{1}{1-z}$ about $z=\frac{1}{4}$ is

1. 1
2. $\frac{1}{4}$
3. $\frac{3}{4}$
4. 0

33. The radius of convergence of the series $\sum_{n=1}^{\infty} \frac{z^n}{n^a}, a \in R$

1. is 1
2. cannot say unless the value of *a* is specified
3. diverges of all value of *a*
4. diverges only for specified values of *a*

34. The radius of convergence of the power series $\sum 5^n z^{3n}$ is

1. 1
2. 5
3. $5^{\frac{1}{3}}$
4. $5^{-\frac{1}{3}}$

35. It is given that $\sum_{n-0}^{\infty} a_n z^n$ converges at $z=3+i4$. Then, the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is

1. ≤ 5
2. ≥ 5
3. < 5
4. > 5

(GATE 2007)

36. The radius of convergence of $\sum_{n=0}^{\infty} \frac{\left(1+\frac{1}{n}\right)^{n^2}}{n^3} z^n$ is

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1. e 2. $\frac{1}{e}$ 3. 1 4. ∞

(GATE 2006)

37. $\int_L |dz|$, where L is any rectifiable arc joining the points $z=a$ and $z=b$ is equal to

1. $b-a$ 2. $|b-a|$
3. arc length of L 4. 0

38. The power series $\sum_{n=0}^{n} 2^{-n} z^{2n}$ converges if

1. $|z| \le 2$ 2. $|z| < 2$
3. $|z| \le \sqrt{2}$ 4. $|z| < \sqrt{2}$

(CSIR NET June 2011)

39. The power series $\sum_{n=0}^{\infty} 3^{-n} (z-1)^{2n}$ converges if

1. $|z| \le 3$ 2. $|z| < \sqrt{3}$
3. $|z-1| < \sqrt{3}$ 4. $|z-1| \le \sqrt{3}$

(CSIR NET June 2012)

40. Consider the power series $\sum_{n=1}^{\infty} z^{n!}$. The radius of convergence of this series is

1. 0
2. ∞
3. 1
4. a real number greater than 1

(CSIR NET Dec 2012)

41. Let $p(x)$ be a polynomial of the real variable x of degree $k \ge 1$. Consider the power series $f(z) = \sum_{n=0}^{\infty} p(n) z^n$ where z is a complex variable. Then the radius of convergence of $f(z)$ is

1. 0 2. 1 3. k 4. ∞

(CSIR NET June 2014)

42. Let $\sum_0^{\infty} a_n z^n$ be a convergent power series such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = R > 0$. Let p be a polynomial of degree d. Then the radius of convergence of the power series $\sum_{n=0}^{\infty} p(n) a_n z^n$ equals

1. $\frac{1}{R}$ 2. d 3. Rd 4. $R+d$

(CSIR NET Dec 2014)

43. Consider the following power series in the complex variable z :

$f(z) = \sum_{n=1}^{\infty} n \log n \, z^n$, $g(z) = \sum_{n=1}^{\infty} \frac{e^{n^2}}{n} z^n$. If r, R are the radii of convergence of f and g respectively, then

1. $r=0, R=1$ 2. $r=1, R=0$
3. $r=1, R=\infty$ 4. $r=\infty, R=1$

(CSIR NET Dec 2015)

44. Let $P(x)$ be a polynomial of degree $d \ge 2$. The radius of convergence of the power series $\sum_{n=0}^{\infty} P(n) z^n$ is :

1. 0 2. 1
3. ∞ 4. dependent on d

(CSIR NET June 2016)

45. Let $f(z)$ be continuous on a contour C of length L and let $|f(z)| \le M$ on C, then $\left| \int_C f(z) dz \right|$ is

1. $\le ML$ 2. $\ge ML$
3. $> ML$ 4. $< ML$

46. Suppose that a function f is continuous in a domain D, then among the following statements

I. f has primitive in D
II. The integral of $f(z)$ along any path lying in D between any two fixed points

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in D is independent of path

III. The integral of $f(z)$ along every closed contour in D is zero.

Codes

1. I implies III but not II
2. II implies III but not implied by I
3. I implies II and III but not implied by either of II or III
4. All the statements are equivalent

47. Which of the following is correct ?
1. Morera's theorem is exact converse of Cauchy's theorem
2. Morera's theorem is valid for all functions $f(z)$ in a domain for which $\int_C f(z)dz = 0$
3. Morera's theorem is short of converse of Cauchy's theorem
4. None of the above

48. The radius of convergence of the power series $f(z) = \sum \frac{n+1}{(n+2)(n+3)} z^n$ is

1. 1 2. 2 3. 3 4. 4

49. The radius of convergence of the power series $\sum_{n=0}^{\infty} (n+2i)^n z^n$ is

1. 0 2. one
3. ∞ 4. None of these

50. The radius of convergence of $\sum_{n=0}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} z^n$ is

1. e 2. $\frac{1}{e}$
3. e^2 4. $\frac{1}{e^2}$

51. If the series $\sum_{n=0}^{\infty} n^2 \left(\frac{z^2+1}{1+i}\right)^n$ converges, then

1. $|z+1| \le \sqrt{2}$ 2. $|z| \le \sqrt{2}$
3. $|z^2+1| < \sqrt{2}$ 4. $|z^2-1| < \sqrt{2}$

52. The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges at

1. $z = 0$ 2. $z = 1$
3. $z = 2$ 4. All of the above

53. The centre of convergence of $\sum_{n=0}^{\infty} (n+2i)^n z^n$ is

1. 0 2. 1
3. 2 4. 3

54. $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$, then

1. for $n \le 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$
2. for $n \ge 0$, $a_n = 0$
3. for $n \ge 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$
4. None of the above

55. If f is analytic in a ball $B(a;R)$ and $|f(z)| \le M, \forall z \in B(a;R)$, then

1. $|f^{(n)}(a)| \le \frac{n!M}{R^n}$ 2. $|f^{(n)}(a)| \ge \frac{n!M}{R^n}$
3. $|f^{(n)}(a)| = n!M$ 4. None of these

56. The series $\sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}, |z| \le 1$ is

1. uniformly but not absolutely convergent
2. uniformly and absolutely convergent
3. absolutely convergent but not uniformly convergent
4. convergent but not uniformly

57. If $f(z)$ is analytic within a circle C, given by $|z-z_0| = R$ and if $|f(z)| \le M$ on C, then $|f^{(n)}(z_0)|$

1. $\ge \frac{n!M}{R^n}$ 2. $> \frac{n!M}{R^n}$
3. $\le \frac{n!M}{R^n}$ 4. None of these

58. A bounded entire function is constant. This statement is
1. Cauchy's theorem
2. Liouville's theorem
3. Morera theorem
4. Schwarz lemma

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59. The theorem if for all z in the entire complex plane, the function $f(z)$ is analytic and bounded, then $f(z)$ must be a constant is named after
    1. Morera 2. Cauchy
    3. Jordan 4. Liouville

60. The radius of convergence of the series $\sum_{n=1}^{\infty} z^{n^2}$ is
    1. 0 2. ∞
    3. 1 4. 2

(CSIR NET Dec 2016)

61. Let C denote the unit circle centered at the origin in $\mathbb{C}$. Then $\frac{1}{2\pi i}\int_C \left|1+z+z^2\right|^2 dz$, where the integral is taken anti-clockwise along C, equals
    1. 0 2. 1
    3. 2 4. 3

(CSIR NET June 2017)

62. Let C be the circle of radius 2 with centre at the origin in the complex plane, oriented in the anti-clockwise direction. Then the integral $\oint_C \frac{dz}{(z-1)^2}$ is equal to
    1. $\frac{1}{2\pi i}$ 2. $2\pi i$
    3. 1 4. 0

(CSIR NET Dec 2017)

63. Let f be a holomorphic function in the open unit disc such that $\lim_{z\to 1} f(z)$ does not exist. Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series of f about $z=0$ and let R be its radius of convergence. Then
    1. $R=0$ 2. $0<R<1$
    3. $R=1$ 4. $R>1$

(CSIR NET Dec 2017)

---------- M C Q ----------

1. Which of the following statements are true ?
    1. If $\{a_k\}$ is bounded then $\sum_0^{\infty} a_k z^k$ defines an analytic function on the open unit disk
    2. If $\sum_0^{\infty} a_k z^k$ defines an analytic function on the open unit disk then $\{a_k\}$ must converge to zero
    3. If $f(z)=\sum_0^{\infty} a_k z^k$ and $g(z)=\sum_0^{\infty} b_k z^k$ are two power series functions whose radii of convergence are 1, then the product $f\cdot g$ has a power series representation of the form $\sum_0^{\infty} c_k z^k$ on the open unit disk
    4. If $f(z)=\sum_0^{\infty} a_k z^k$ has a radius of convergence 1, then f is continuous on $\Omega=\{z\in\mathbb{C}\,|\,|z|\le 1\}$

(CSIR NET June 2018)

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True false key

1. T	2. T	3. F	4. T
5. T	6. T	7. T	8. T
9. T	10. T	11. T	12. T
13. T	14. T	15. T	16. F
17. T	18. T	19. T	20. T
21. T			

Assignment key

S C Q

1. 2	2. 3	3. 2	4. 1
5. 1	6. 4	7. 1	8. 3
9. 1	10. 2	11. 3	12. 1
13. 2	14. 1	15. 2	16. 2
17. 3	18. 2	19. 3	20. 1
21. 3	22. 1	23. 4	24. 2
25. 1	26. 1	27. 2	28. 2
29. 3	30. 3	31. 2	32. 3
33. 1	34. 4	35. 2	36. 2
37. 3	38. 4	39. 3	40. 3
41. 2	42. 1	43. 2	44. 2
45. 1	46. 4	47. 3	48. 1
49. 1	50. 2	51. 3	52. 4
53. 1	54. 3	55. 1	56. 2
57. 3	58. 2	59. 4	60. 3
61. 3	62. 4	63. 3	

M C Q

1. 1,3

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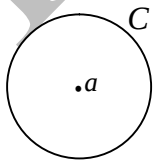
Chapter - 4

Calculus of Residues

4.1 Expansions

Taylor's theorem : Let $f(z)$ be an analytic function inside a circle C with centre at a, then for all z in C,

$$f(z)=\sum_{n=0}^{\infty} f^{(n)}(a)\frac{(z-a)^n}{n!}= f(a)+f'(a)(z-a)+\frac{f''(a)(z-a)^2}{2!}+\frac{f'''(a)(z-a)^3}{3!}+.....$$

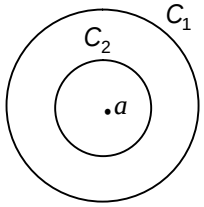


Corollary : Let $f(z)$ be analytic inside a circle C with centre at a, then f has a Maclaurin series expansion

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n = f(0)+f'(0)z+\frac{f''(0)}{2!}z^2+\frac{f'''(0)}{3!}z^3+....$$

Laurent's Theorem : Let $f(z)$ be analytic inside and on the boundary of the ring shaped region R bounded by two concentric circles C_1 and C_2 with centre at 'a' and radii r_1 and r_2 ($r_2 < r_1$) respectively, then for all z in R,

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty} b_n(z-a)^{-n}$$



where $a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-a)^{n+1}}dz$, $n = 0, 1, 2,\ldots$

and $b_n=\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-a)^{-n+1}}dz$, $n = 1, 2, 3, \ldots$

Remarks : (1) The above series is called Laurent's series of function $f(z)$ about the point 'a'. This series can be put in more compact form as : $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ where $a_n=\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-a)^{n+1}}dz$

where γ denotes C_1 for $n\geq 0$ and C_2 for $n<0$.

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(2) In the above Laurent's series, the series of negative powers of $(z-a)$ i.e. $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is called the principal part of Laurent's expansion of $f(z)$. On the other hand, the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called regular part.

(3) Some familiar expansions:

(i) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots$ valid for all z

(ii) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$ valid for all z

(iii) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$ valid for all z

(iv) $\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \ldots..$ valid for all z

(v) $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots.$ valid for all z

(vi) $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots$ valid for all z

(viii) $\tanh z = z - \frac{1}{3}z^3 + \frac{2}{15}z^5 - \frac{17}{315}z^7 + \ldots..$ valid for all z

(viii) $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots$ valid for $|z| < 1$

(ix) $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$ valid for $|z| < 1$

(x) $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots.$ valid for $|z| < 1$

(xi) $\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \ldots.$ valid for $|z| < 1$

Exercise 4.1

In problems 1–6, expand the given function in a Laurent series valid for the given annular domain.

1. $f(z) = \frac{\cos z}{z}, 0 < |z|$

2. $f(z) = \frac{z - \sin z}{z^5}, 0 < |z|$

3. $f(z) = e^{-\frac{1}{z^2}}, 0 < |z|$

4. $f(z) = \frac{1 - e^z}{z^2}, 0 < |z|$

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5. $f(z)=\dfrac{e^z}{z-1}, 0<|z-1|$

6. $f(z)=z\cos\dfrac{1}{z}, 0<|z|$

In problem 7–12, expand $f(z)=\dfrac{1}{z(z-3)}$ in a Laurent series valid for the indicated annular domain.

7. $0<|z|<3$

8. $|z|>3$

9. $0<|z-3|<3$

10. $|z-3|>3$

11. $1<|z-4|<4$

12. $1<|z+1|<4$

In problem 13–16, expand $f(z)=\dfrac{1}{(z-1)(z-2)}$ in a Laurent series valid for the given annular domain.

13. $1<|z|<2$

14. $|z|>2$

15. $0<|z-1|<1$

16. $0<|z-2|<1$

In problem 17–20, expand $f(z)=\dfrac{z}{(z+1)(z-2)}$ in a Laurent series valid for the given annular domain.

17. $0<|z+1|<3$

18. $|z+1|>3$

19. $1<|z|<2$

20. $0<|z-2|<3$

In problem 21 and 22, expand $f(z)=\dfrac{1}{z(1-z)^2}$ in a Laurent series valid for the given annular domain.

21. $0<|z|<1$

22. $|z|>1$

In problem 23 and 24, expand $f(z)=\dfrac{1}{(z-2)(z-1)^3}$ in a Laurent series valid for the given annular domain.

23. $0<|z-2|<1$

24. $0<|z-1|<1$

In problem 25 and 26, expand $f(z)=\dfrac{7z-3}{z(z-1)}$ in a Laurent series valid for the given annular domain.

25. $0<|z|<1$

26. $0<|z-1|<1$ $\left[\text{Hint:}\dfrac{7z-3}{z}=\dfrac{7(z-1)+4}{1+(z-1)}\right]$

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In problem 27 and 28, expand $f(z)=\dfrac{z^2-2z+2}{z-2}$ in a Laurent series valid for the given annular domain.

27. $1<|z-1|$

28. $0<|z-2|$

In problem 29 to 32, use $\cos z=1-\dfrac{z^2}{2!}+\dfrac{z^4}{4!}-...,\sin z=z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-...,$ and long division to find the first three nonzero terms of a Laurent series of the given function f valid for $0<|z|<\pi$.

29. $f(z)=\operatorname{cosec} z$

30. $f(z)=\cot z$

31. $f(z)=\tan z$

32. $f(z)=\sec z$

In problem 33 to 36, use $\cosh z=1+\dfrac{z^2}{2!}+\dfrac{z^4}{4!}+...,\sinh z=z+\dfrac{z^3}{3!}+\dfrac{z^5}{5!}+...,$ and long division to find the first three nonzero terms of a Laurent series of the given function f valid for $0<|z|<\pi$.

33. $f(z)=\operatorname{cosech} z$

34. $f(z)=\coth z$

35. $f(z)=\tanh z$

36. $f(z)=\operatorname{sech} z$

Answers

1. $f(z)=\dfrac{1}{z}-\dfrac{z}{\lfloor 2}+\dfrac{z^3}{\lfloor 4}-\dfrac{z^5}{\lfloor 6}+....$

2. $f(z)=\dfrac{1}{\lfloor 3 z^2}-\dfrac{1}{\lfloor 5}+\dfrac{z^2}{\lfloor 7}-....$

3. $f(z)=1-\dfrac{1}{\lfloor 1 z^2}+\dfrac{1}{\lfloor 2\cdot z^4}-\dfrac{1}{\lfloor 3\cdot z^6}+....$

4. $f(z)=-\dfrac{1}{z}-\dfrac{1}{\lfloor 2}-\dfrac{z}{\lfloor 3}-....$

5. $f(z)=\dfrac{e}{z-1}+e+\dfrac{e(z-1)}{\lfloor 2}+\dfrac{e(z-1)^2}{\lfloor 3}+....$

6. $f(z)=z-\dfrac{1}{2!z}+\dfrac{1}{4!z^3}-....$

7. $f(z)=-\dfrac{1}{3z}-\dfrac{1}{3^2}-\dfrac{z}{3^3}-\dfrac{z^2}{3^4}-....$

8. $f(z)=\dfrac{1}{z^2}+\dfrac{3}{z^3}+\dfrac{9}{z^4}+\dfrac{27}{z^5}+.....$

9. $f(z)=\dfrac{1}{3(z-3)}-\dfrac{1}{3^2}+\dfrac{z-3}{3^3}-\dfrac{(z-3)^2}{3^4}+....$

10. $f(z)=\dfrac{1}{(z-3)^2}-\dfrac{3}{(z-3)^3}+\dfrac{9}{(z-3)^4}-....$

11. $f(z)=.....-\dfrac{1}{3(z-4)^2}+\dfrac{1}{3(z-4)}-\dfrac{1}{12}+\dfrac{z-4}{3\cdot 4^2}-\dfrac{(z-4)^2}{3\cdot 4^3}+.....$

12. $f(z)=.....-\dfrac{1}{3(z+1)^3}-\dfrac{1}{3(z+1)^2}-\dfrac{1}{3(z+1)}-\dfrac{1}{12}-\dfrac{1}{4\cdot 12}(z+1)-\dfrac{1}{12\cdot 4^2}(z+1)^2-.....$

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13. $f(z)=.....-\frac{1}{z^2}-\frac{1}{z}-\frac{1}{2}-\frac{z}{2^2}-\frac{z^2}{2^3}-.....$

14. $f(z)=\frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+....$

15. $f(z)=-\frac{1}{z-1}-1-(z-1)-(z-1)^2-....$

16. $f(z)=\frac{1}{z-2}-1+(z-2)-....$

17. $f(z)=\frac{1}{3(z+1)}-\frac{2}{3^2}-\frac{2(z+1)}{3^3}-\frac{2(z+1)^2}{3^4}-...$

18. $f(z)=\frac{1}{z+1}+\frac{2}{(z+1)^2}+\frac{2\cdot 3}{(z+1)^3}+\frac{2\cdot 3^2}{(z+1)^4}+....$

19. $f(z)=....-\frac{1}{3z^2}+\frac{1}{3z}-\frac{1}{3}-\frac{z}{3\cdot 2}-\frac{z^2}{3\cdot 2^2}-.....$

20. $f(z)=\frac{2}{3(z-2)}+\frac{1}{3^2}-\frac{(z-2)}{3^3}+\frac{(z-2)^2}{3^4}-....$

21. $f(z)=\frac{1}{z}+2+3z+4z^2+.....$

22. $f(z)=\frac{1}{z^3}+\frac{2}{z^4}+\frac{3}{z^5}+.....$

23. $f(z)=\frac{1}{z-2}-3+6(z-2)-10(z-2)^2+.....$

24. $f(z)=-\frac{1}{(z-1)^3}-\frac{1}{(z-1)^2}-\frac{1}{(z-1)}-1-(z-1)-.....$

25. $f(z)=\frac{3}{z}-4-4z-4z^2-.....$

26. $f(z)=3\left[1-(z-1)+(z-1)^2-....\right]+\frac{4}{z-1}$

27. $f(z)=.....+\frac{2}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2}{z-1}+1+(z-1)$

28. $f(z)=\frac{2}{z-2}+2+(z-2)$

29. $f(z)=\frac{1}{z}+\frac{z}{6}+\frac{7z^3}{360}+.....$

30. $f(z)=\frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-.....$

31. $f(z)=z+\frac{z^3}{3}+\frac{2}{15}z^5+.....$

32. $f(z)=1+\frac{z^2}{2}+\frac{5}{24}z^4+.....$

33. $f(z)=\frac{1}{z}-\frac{z}{6}+\frac{7}{360}z^3-.....$

34. $f(z)=\frac{1}{z}+\frac{z}{3}-\frac{1}{45}z^3+.....$

35. $f(z)=z-\frac{z^3}{3}+\frac{2}{15}z^5-.....$

36. $1-\frac{z^2}{2}+\frac{5}{24}z^4$

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4.2 Zeros and their order

Def. Zeros : A number z_0 is zero of a function f if $f(z_0)=0$. We say that an analytic function f has a zero of order n at $z=z_0$ if $f(z_0)=0, f'(z_0)=0, f''(z_0)=0,..., f^{(n-1)}(z_0)=0,$ but $f^{(n)}(z_0)\neq 0$. A zero of order n is also referred to as a zero of multiplicity n. A zero of order one is called a simple zero.

Theorem : Zero of order n : A function f that is analytic in some disk $|z-z_0|<R$ has a zero of order n at $z=z_0$ if and only if f can be written as $f(z)=(z-z_0)^n\phi(z)$, where ϕ is analytic at $z=z_0$ and $\phi(z_0)\neq 0$.

Exercise 4.2

In problem 1–6, determine the zeros and their order for the given function.

1. $f(z)=(z+2-i)^2$
2. $f(z)=z^4-16$
3. $f(z)=z^4+z^2$
4. $f(z)=\sin^2 z$
5. $f(z)=e^{2z}-e^z$
6. $f(z)=ze^z-z$

In problems 7–18, the indicated number is a zero of the given function. Use a Maclarin or Taylor series to determine the order of the zero.

7. $f(z)=z(1-\cos^2 z); z=0$
8. $f(z)=z-\sin z; z=0$
9. $f(z)=1-e^{z-1}; z=1$
10. $f(z)=1-\pi i+z+e^z; z=\pi i$
11. $f(z)=\sin z^2; z=0$
12. $f(z)=\sin^2 z^2; z=0$
13. $f(z)=\sin^2 z(1-\cos^2 z); z=0$
14. $f(z)=1-\cos^4 z; z=0$
15. $f(z)=1-\cos^2 z^2; z=0$
16. $f(z)=z^3-\sin^3(z);\ z=0$
17. $f(z)=\sin^m(z^n),\ m,n\in\mathbb{Z}^+;\ z=0$
18. $f(z)=1-\cos z\ ;\ z=0$

Answers

1. $-2+i$ is a zero of order 2.
2. $\pm 2, \pm 2i$ are simple zeros.
3. 0 is a zero of order 2, i and $-i$ are simple zeros.
4. $n\pi,\ n=0,\pm 1,....$ are zeros each of order 2.
5. $2n\pi i,\ n=0,\pm 1,.....$ are simple zeros.
6. 0 is a zero of order 2, $2n\pi i,\ n=\pm 1,\pm 2,.....$ are simple zeros.

7. 3 8. 3 9. 1 10. 2 11. 2 12. 4 13. 4 14. 2
15. 4 16. 5 17. mn 18. 2

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4.3 Zeros are isolated

Def. Isolated zero : A zero $z = z_0$ of a function f is said to be isolated if there exists $r > 0$ such that the disc $|z - z_0| < r$ does not contain any other zero of the function f.

Def. Non-isolated zero : A zero $z = z_0$ of a function f is said to be non-isolated if every disc centered at z_0 contains atleast one zero of f other than z_0.

Results :

1. If a function has only finitely many zeros then all of them are isolated.
2. All zeros of a polynomial or rational function are isolated.
3. If a function has infinitely many zeros then they may or may not be isolated. For example, the functions $\sin z, \cos z, \sinh z, \cosh z$ has infinitely many zeros and all of them are isolated. On the other hand consider the function $f(z) = \begin{cases} \sin\left(\frac{1}{z}\right) & z \neq 0 \\ 0 & z = 0 \end{cases}$. The zero $z = 0$ is non-isolated and the zeros $z = \frac{1}{n\pi}$, $n \neq 0$, all are isolated.
4. The identically zero function defined on a domain has all non-isolated zeros.
5. **Theorem : Zeroes are isolated :**

Let $f \not\equiv 0$ be an analytic function in a domain D then every zero of $f(z)$ in D is isolated.

or

Let $f \not\equiv 0$ be an analytic function in a domain D and z_0 be a zero of $f(z)$ then their exists a $r > 0$ such that $f(z)$ has no zero other than z_0 in the disc $|z - z_0| < r$.

or

Let $f \not\equiv 0$ be an analytic function defined on a domain D, then the set of zeros of f cannot have a limit point in D, i.e., if S is the set of zeros of f then $S' \cap D = \phi$.

6. **Identity theorem and its different forms :**

(i) Let $f(z)$ be an analytic function in a domain D and there is a zero z_0 of $f(z)$ inside D which is not isolated then $f(z) \equiv 0$ *i.e.* $f(z) = 0$ for all z in D.

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(ii) If f is an analytic function defined on a domain D and the set of zeros of $f(z)$ in D has a limit point in D, then $f \equiv 0$, i.e., $f(z) = 0$ for all $z \in D$.

(iii) Let f be an analytic function defined on a domain D and let S be any subset of D having a limit point in D. If $f(z) = 0$ for all $z \in S$, then $f(z) = 0$ for all $z \in D$ i.e., $f \equiv 0$.

(iv) Let f and g be analytic functions defined on a domain D and let S be any subset of D having a limit point in D. If $f(z) = g(z)$ for all $z \in S$, then $f(z) = g(z)$ for all $z \in D$ i.e., $f \equiv g$.

(v) If $f(z)$ is an analytic function in a domain D and PQ is an arc inside D such that $f(z) = 0$ for all $z \in PQ$ then $f(z) = 0$ for all z in D.

(vi) In above result, arc PQ can be replaced by a region R.

Theorem : Let D be a domain and $z_0 \in D$. Let f is analytic on $D - \{z_0\}$ i.e., f may or may not be analytic at z_0. Let $\langle z_n \rangle$ be a sequence in D such that $f(z_n) = 0$ for all n and $z_n \to z_0$, then

(i) If $f(z)$ is analytic at z_0 then $f \equiv 0$.

(ii) If $f(z)$ is not analytic at z_0 then z_0 is an isolated essential singularity of $f(z)$.

Example : $f(z) = \begin{cases} \sin\frac{1}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$, $z = 0$ is a zero of $f(z)$ and also it is an isolated essential singularity.

Exercise 4.3

1. Let f be an analytic function defined on open unit disc. Let S be the set of zeros of f. If S contains any of the following sets then show that $f \equiv 0$.

(i) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ (ii) $\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}$

(iii) $\left\{\frac{1}{2n+1} : n \in \mathbb{N}\right\}$ (iv) $\left\{\frac{1}{4n+7} : n \in \mathbb{N}\right\}$

(v) $\left\{\frac{1}{2} + \frac{1}{n} : n \in \mathbb{N}\right\}$ (vi) $\left\{-\frac{1}{3} + \frac{1}{n} : n \in \mathbb{N}\right\}$

(vii) $\left\{\frac{1}{2}i + \frac{1}{n}i : n \in \mathbb{N}\right\}$ (viii) $\{0\} \cup \left\{\frac{1}{4n+7} : n \in \mathbb{N}\right\}$

(ix) $\left\{z \in D : |z| = \frac{1}{3}\right\}$ (x) $\{z \in D : \mathrm{Im}(z) = 0\}$

2. Let $D = \{z : |z| < 1\}$ and $H(D)$ denotes the set of all holomorphic functions on D. Suppose

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$F = \left\{ f \in H(D) : f\left(\frac{1}{2n+1}\right) = 0 \ \forall \ n \geq 1 \right\}$. Then what is $\text{card}(F)$.

3. What is the number of analytic functions defined on the open unit disc which vanish on the set $S = \{0\} \cup \left\{ \frac{1}{2n+1} : n \in \mathbb{N} \right\}$.

4. What is the number of analytic functions defined on the open unit disc which vanish only on the set $S = \left\{ \frac{1}{2n+1} : n \in \mathbb{N} \right\}$.

5. Let f be an analytic function defined on the open unit disc such that $f\left(\frac{1}{n}\right) = \frac{2n}{3+4n}$ for all $n \geq 1$, then find f.

6. Let f be an analytic function defined on the open unit disc such that $f\left(\frac{1}{n^2}\right) = \frac{n^2}{4+5n^2}$ for all $n \geq 1$, then find f.

7. Show that there is no analytic function defined on the open unit disc such that $f\left(\frac{1}{2n}\right) = \frac{2n}{3+4n}$ for all $n \geq 1$.

8. Let f be an analytic function defined on the open unit disc such that $f\left(1-\frac{1}{n}\right) = \left(1-\frac{1}{n}\right)^2$ for all $n \geq 1$, then can we conclude that $f(z) = z^2$ for all $|z| < 1$.

9. Let f be an analytic function defined on the open disc $|z| < 2$ such that $f\left(1-\frac{1}{n}\right) = \left(1-\frac{1}{n}\right)^2$ for all $n \geq 1$, then can we conclude that $f(z) = z^2$ for all $|z| < 2$.

10. Show that there is no analytic function defined on the open unit disc such that

(i) $f\left(\frac{1}{2n}\right) = \frac{1}{2n}$ and $f\left(\frac{1}{2n+1}\right) = 1 + \frac{1}{2n+1}$ for all $n \geq 1$.

(ii) $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$ for all $n \geq 1$.

(iii) $f\left(\frac{1}{2}\right) = \frac{1}{2}, f\left(\frac{1}{4}\right) = \frac{1}{4}, \ldots$ and $f\left(\frac{1}{3}\right) = \frac{1}{2}, f\left(\frac{1}{5}\right) = \frac{1}{4}, \ldots$

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11. Let f and g are two analytic functions defined on a domain D such that $f(z)\cdot g(z)=0$ for all $z\in D$ then prove that either $f(z)=0$ for all $z\in D$ or $g(z)=0$ for all $z\in D$. In words, we can say that, if product of two analytic functions inside a domain D is zero then one of them must be identically zero.
12. If $H(D)$ denotes the collection of all holomorphic functions defined on a domain D then show that $H(D)$ is an integral domain with respect to pointwise addition and multiplication of the functions.

4.4 Singularities in finite complex plane

Def. Singularity or Singular Point : If a complex function f fails to be analytic at a point $z=z_0$, then this point is said to be a singularity or singular point of the function.

Def. Isolated singularity : A singularity $z=z_0$ of a function f is called isolated singularity if there exists a $r>0$ such that the disc $|z-z_0|<r$ does not contain any singularity of f other than z_0.

Def. Non-Isolated Singularity : A singularity $z=z_0$ of a function f is called non-isolated singularity if every disc centered at z_0 contains atleast one singularity of f other than z_0.

Def. Types of Isolated singularities : The isolated singularities of a function are of three types :

(i) Removable Singularity (ii) Pole (iii) Isolated Essential Singularity

Classification : Let $z=z_0$ be an isolated singularity of $f(z)$. Then by definition, there exists a deleted neighborhood, of z_0 in which $f(z)$ is analytic. For any point z of this neighborhood, by Laurent's expansion, we have

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}b_n(z-z_0)^{-n}$$

where a_n and b_n are given as usual. Here $\sum_{n=1}^{\infty}b_n(z-z_0)^{-n}$ is the principal part of expansion of $f(z)$.

Now, we give the following definitions :

Def. Removable singularity : If the principal part of $f(z)$ contains no term (i.e. there is no negative powers of $(z-z_0)$ in the expansion of $f(z)$), then $z=z_0$ is called a removable singularity.

e.g. $$\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-.....\right)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-.....$$

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Since there is no negative power of z. Hence $z=0$ is a removable singularity of $f(z)$.

Def. Pole : If the principal part contains a finite number of terms then the singularity $z=z_0$ is called a pole of $f(z)$. Further, if $b_m \neq 0$ and $b_n = 0$ for all $n > m$, then $z=z_0$ is called a pole of order *m*.

Theorem : Pole of order *m* : A function *f* analytic in a punctured disc $0<|z-z_0|<R$ has a pole of order *m* at $z=z_0$ iff *f* can be written $f(z)=\dfrac{\phi(z)}{(z-z_0)^m}$, where ϕ is analytic at $z=z_0$ and $\phi(z_0)\neq 0$.

Proof : If $z=z_0$ is a pole of order *m*, then $f(z)$ has an expansion of the form :

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} b_n (z-z_0)^{-n}, \text{ where } b_m \neq 0$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n + b_1 (z-z_0)^{-1} + b_2 (z-z_0)^{-2} + \ldots\ldots\ldots + b_m (z-z_0)^{-m}$$

$$= \frac{1}{(z-z_0)^m}\left[\sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2} + \ldots\ldots\ldots + b_m\right]$$

$$= \frac{\phi(z)}{(z-z_0)^m} \quad \text{where } \phi(z) \text{ is analytic and } \phi(z_0)=b_m \neq 0.$$

Def. Essential singularity : If the principal part contains an infinite number of terms i.e. $b_n \neq 0$ for infinitely many values of *n*, then the singularity $z=z_0$ is called an essential singularity.

e.g. $e^{1/z} = 1 + \dfrac{1}{z} + \dfrac{1}{2!\,z^2} + \ldots\ldots\ldots$

There are infinitely many negative powers of *z*, so $z=0$ is an essential singularity of $e^{1/z}$.

Another way of defining various isolated singularities : Let $z=z_0$ is an isolated singularity of a function $f(z)$, then

(i) $z=z_0$ is removable singularity of $f(z)$ iff $\lim\limits_{z\to z_0} f(z)$ exist finitely.

(ii) $z=z_0$ is pole of $f(z)$ iff $\lim\limits_{z\to z_0} f(z)=\infty$. Further, if $z=z_0$ is a pole of $f(z)$ and

$\lim\limits_{z\to z_0}(z-z_0)^{n-1} f(z)=\infty$ and $\lim\limits_{z\to z_0}(z-z_0)^{n} f(z)\neq 0, \infty$, then *n* is the order of the pole $z=z_0$, i.e.,

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n is the smallest positive integer such that $\lim_{z\to z_0}(z-z_0)^n f(z)\neq 0,\infty$. Also if $z=z_0$ is a pole of order n and k is a positive integer such that $k>n$ then $\lim_{z\to z_0}(z-z_0)^k f(z)=0$.

(iii) $z=z_0$ is essential singularity iff $\lim_{z\to z_0} f(z)$ does not exist finitely or infinitely.

Results :

1. Limit point of zeros is isolated essential singularity.
2. Limit point of poles is non-isolated essential singularity.
3. Limit point of essential singularities is a non-isolated essential singularity.

Result : (i) If $z=z_0$ is a removable singularity of a function $f(z)$ then $z=z_0$ is also a removable singularity of the functions $e^{f(z)}$, $\sin(f(z))$, $\cos(f(z))$, $\sinh(f(z))$ and $\cosh(f(z))$.

(ii) If $z=z_0$ is a pole of a function $f(z)$ then $z=z_0$ is an essential singularity of the functions $e^{f(z)}$, $\sin(f(z))$, $\cos(f(z))$, $\sinh(f(z))$ and $\cosh(f(z))$.

(iii) If $z=z_0$ is an essential singularity of a function $f(z)$ then $z=z_0$ is an essential singularity of the functions $e^{f(z)}$, $\sin(f(z))$, $\cos(f(z))$, $\sinh(f(z))$ and $\cosh(f(z))$.

Def. Meromorphic function : A function f is meromorphic if it is analytic throughout a domain D, except possibly for poles in D. e.g. $f(z)=\dfrac{1}{z^2+1}$

Result : If $f(z)=\dfrac{p(z)}{q(z)}$ is a rational function and $z=z_0$ is a singularity of $f(z)$ and let $f(z)=\dfrac{(z-z_0)^m\,\phi(z)}{(z-z_0)^n\,\psi(z)}$, where $\phi(z_0)\neq 0$, $\psi(z_0)\neq 0$ then

(i) If $m\geq n$ then $z=z_0$ is removable singularity.

(ii) If $m<n$ then $z=z_0$ is a pole of order $n-m$.

Result : A rational function never has an essential singularity i.e. all the singularities of a rational function are either removable or poles.

Riemann's theorem on removable singularity : Let D be a domain (i.e., open and connected set) and $z_0\in D$. Suppose $f(z)$ is analytic on $D-\{z_0\}$, z_0 is an singularity of $f(z)$ and $f(z)$ is bounded on D then z_0 is a removable singularity of $f(z)$.

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Riemann's removability theorem : Suppose that f is continuous on a domain D and analytic on $D-\{z_0\}$ for some $z_0 \in D$. Then f is analytic on D.

Exercise 4.4

Determine singularities in C of the following :

Type I

1. $f(z)=\dfrac{e^z-1}{e^z}$
2. $f(z)=\dfrac{e^z+1}{e^z}$
3. $f(z)=\dfrac{e^z}{e^z+1}$
4. $f(z)=\dfrac{e^z}{e^z-1}$
5. $f(z)=\dfrac{e^z-1}{z(z-2)}$
6. $f(z)=\dfrac{e^z}{z^2+4}$
7. $f(z)=\dfrac{1-e^z}{1+e^z}$
8. $f(z)=\dfrac{e^z+1}{e^z-1}$
9. $f(z)=\dfrac{1}{z\left(e^z-1\right)}$
10. $f(z)=\dfrac{1}{z^2\left(e^z-1\right)}$
11. $f(z)=\dfrac{e^z}{1+z^2}$
12. $f(z)=\dfrac{1+z^2}{e^z}$
13. $f(z)=\left(\dfrac{e^z-1}{z}\right)^2$
14. $f(z)=\dfrac{z}{e^z-1}$
15. $f(z)=\dfrac{\sin z}{z}$
16. $f(z)=\dfrac{\cot \pi z}{(z-a)^2}$; $a \notin \mathbb{Z}$
17. $f(z)=\dfrac{1}{\sin z-\sin a}$
18. $f(z)=z\operatorname{cosec} z$
19. $f(z)=\dfrac{z-\sin z}{z^3}$
20. $f(z)=\dfrac{1}{\cos z+\cos a}$
21. $f(z)=\dfrac{\cot z}{z^2}$
22. $f(z)=\dfrac{\cos z}{z^2}$
23. $f(z)=\dfrac{1}{z^3(2-\cos z)}$
24. $f(z)=\dfrac{2(1-\cos z)}{z^2}$
25. $f(z)=\dfrac{z-\sin z}{z\sin z}$
26. $f(z)=\left(1-z^2\right)\operatorname{cosec}(\pi z)$
27. $f(z)=\dfrac{z\cos\left(\dfrac{\pi z}{2a}\right)}{(z-a)\left(z^2+b^2\right)\sin^5 z}$, $a \neq n\pi$, $n \in \mathbb{Z}$ and $b \neq 0$.
28. $f(z)=\dfrac{\sin^4 z}{z^8}$

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29. $f(z)=\dfrac{(1+z)\cos z}{z}$

30. $f(z)=\operatorname{cosec} z-\dfrac{1}{z}$

31. $f(z)=\dfrac{z^8+z^4+2}{(z-1)^3(3z+2)^2}$

32. $f(z)=\dfrac{z^2+a^2}{z+ia}$

33. $f(z)=\dfrac{z^8}{(z-1)^2}$

34. $f(z)=\dfrac{1-8z^3}{1-4z^2}$

35. $f(z)=\dfrac{z^2}{(z-ip)^4(z+q)^3}$, $p,q\in R-\{0\}$

Type II

36. $f(z)=\dfrac{e^{\frac{1}{z-1}}}{e^z-1}$

37. $f(z)=e^{\frac{z}{\sin z}}$

38. $f(z)=\dfrac{e^{c/z-a}}{e^{\frac{z}{a}-1}}$

39. $f(z)=e^{\frac{z}{1-\cos z}}$

40. $f(z)=e^{-\frac{1}{z^2}}$

41. $f(z)=\dfrac{1}{z}e^{\frac{1}{z}}$

42. $f(z)=ze^{\frac{1}{z^2}}$

43. $f(z)=\dfrac{e^{1/z}}{z^2+4}$

44. $f(z)=(z-3)\sin\left(\dfrac{1}{z+2}\right)$

45. $f(z)=\dfrac{z-2}{z^2}.\sin\left(\dfrac{1}{z-1}\right)$

46. $f(z)=(z-4)\sin\left(\dfrac{1}{z-4}\right)$

47. $f(z)=(z-a)^{2014}\sin\left(\dfrac{1}{z-2015}\right)$

48. $f(z)=\dfrac{\sin\left(\dfrac{1}{z}\right)}{z^2-4}$

49. $f(z)=\sin\left(\dfrac{1}{z}\right)+\dfrac{1}{z^2(2-z)}$

50. $f(z)=\sin\left(1-\dfrac{1}{z}\right)$

51. $f(z)=\dfrac{\sin(\pi z)}{(z-1)}\dfrac{e^{1/z^3}}{z^3}$

52. $f(z)=\dfrac{z^{10}\left(z^2+1\right)e^{\frac{1}{(z-4)^5}}}{(z-\pi)^2\sin^{10}z}$

Type III

53. $f(z)=e^{-z}\sec\dfrac{1}{z}$

54. $f(z)=\dfrac{1}{z^2\sin\left(\dfrac{1}{z}\right)}$

55. $f(z)=\tan\left(\dfrac{1}{z}\right)$

Answers

1. No singularity
2. No singularity
3. $z=(2n+1)\pi i, n\in\mathbb{Z}$ are simple poles

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4. $z = 2n\pi i,\ n \in \mathbb{Z}$ are simple poles
5. $z = 0$ is the removable singularity and $z = 2$ is the simple pole
6. $z = \pm 2i$ are the simple poles
7. $z = (2n+1)\pi i,\ n \in \mathbb{Z}$ are the simple poles
8. $z = 2n\pi i,\ n \in \mathbb{Z}$ are the simple poles
9. $z = 0$ is the pole of order 2 and $z = 2n\pi i, n = \pm 1, \pm 2, \ldots.$ are the simple poles
10. $z = 0$ is the pole of order 3 and $z = 2n\pi i,\ n = \pm 1, \pm 2, \ldots.$ are the simple poles
11. $z = \pm i$ are the simple poles
12. No singularity
13. $z = 0$ is the removable singularity
14. $z = 0$ is the removable singularity and $z = 2n\pi i,\ n = \pm 1, \pm 2, \ldots..$ are the simple poles
15. $z = 0$ is the removable singularity
16. $z = n,\ n \in \mathbb{Z}$ are the simple poles and $z = a,\ a \notin \mathbb{Z}$ is the pole of order 2
17. $z = \{2n\pi + a\},\ \{(2n+1)\pi - a\},\ n \in \mathbb{Z}$ are the simple poles.
18. $z = 0$ is the removable singularity and $z = n\pi,\ n = \pm 1, \pm 2, \ldots.$ are the simple poles.
19. $z = 0$ is the removable singularity
20. $z = (2n+1)\pi \pm a,\ n \in \mathbb{Z}$ is the simple pole
21. $z = 0$ is the pole of order 3 and $z = n\pi,\ n = \pm 1, \pm 2, \ldots.$ are the simple poles
22. $z = 0$ is the pole of order 2
23. $z = 0$ is the pole of order 3 and the roots of $\cos z = 2$ are the simple poles.
24. $z = 0$ is the removable singularity
25. $z = 0$ is the removable singularity and $z = n\pi,\ n = \pm 1, \pm 2, \ldots..$ are the simple poles
26. $z = \pm 1$ are the removable singularities and $z = n,\ n = 0, \pm 2, \pm 3, \ldots.$ are the simple poles
27. $z = 0$ is the pole of order 4 and $z = a$ is the removable singularity, $z = \pm ib$ are the simple poles and $z = n\pi,\ n = \pm 1, \pm 2, \ldots$ are the poles of order 5.
28. $z = 0$ is the pole of order 4
29. $z = 0$ is the simple pole
30. $z = 0$ is the removable singularity and $z = n\pi,\ n = \pm 1, \pm 2, \ldots..$ are the simple poles

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31. $z=1$ is the pole of order 3 and $z=-\frac{2}{3}$ is the pole of order 2

32. $z=-i\,a$ is the removable singularity

33. $z=1$ is the pole of order 2

34. $z=\frac{1}{2}$ is the removable singularity and $z=-\frac{1}{2}$ is the simple pole

35. $z=i\,p$ is the pole of order 4 and $z=-q$ is the pole of order 3

36. $z=1$ is the essential singularity and $z=2n\pi\,i,\ n=0,-1,\pm2,....$ the simple poles

37. $z=0$ is the removable singularity and $z=n\pi,\ n=\pm1,\pm2,....$ are the essential singularities

38. $z=a$ is the essential singularity

39. $z=2n\pi,\ n\in\mathbb{Z}$ are the essential singularities

40. $z=0$ is the essential singularity

41. $z=0$ is the essential singularity

42. $z=0$ is the essential singularity

43. $z=0$ is the essential singularity and $z=\pm2i$ are the simple poles

44. $z=-2$ is the essential singularity

45. $z=0$ is the pole of order 2 and $z=1$ is the essential singularity

46. $z=4$ is the essential singularity

47. $z=2015$ is the essential singularity

48. $z=0$ is the essential singularity and $z=\pm2$ are the simple poles

49. $z=0$ is the essential singularity and $z=2$ is the simple pole

50. $z=0$ is the essential singularity

51. $z=0$ is the essential singularity and $z=1$ is the removable singularity

52. $z=0$ is the removable singularity, $z=\pi$ is the pole of order 12 , $z=4$ is the essential singularity and $z=-\pi,\pm2\pi,\pm3\pi,....$ are the poles of order 10.

53. $z=0$ is the non-isolated essential singularity and $z=\frac{2}{(2n+1)\pi},n\in\mathbb{Z}$, are the simple poles.

54. $z=0$ is the non-isolated essential singularity ; $z=\frac{1}{n\pi},n=\pm1,\pm2,\pm3,....$ are simple poles.

55. $z=0$ is the non-isolated essential singularity and $z=\frac{2}{(2n+1)\pi},n\in\mathbb{Z}$, are the simple poles.

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4.5 Behaviour at infinity

Def. Behaviour At $z=\infty$: The behavior of a function $f(z)$ at $z=\infty$ is defined to be same as that of the function $f\left(\frac{1}{z}\right)$ at $z=0$.

Note : $\mathbb{C}$ = Complex plane or finite complex plane and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ = Extended complex plane.

Results :

1. The constant function is analytic at $z=\infty$.
2. Constant function is the only function which has no singularity in C_∞.
3. A polynomial of degree *k* has a pole of order *k* at the point $z=\infty$.
4. If $f(z)=\frac{p(z)}{q(z)}$ is a rational function where $\deg(p(z))=m$ and $\deg(q(z))=n$, then at the point $z=\infty$, we have the following behavior:
   (i) If $m \le n$ then $z=\infty$ is removable singularity.
   (ii) If $m > n$ then $z=\infty$ is a pole of order $m-n$.

Def : Transcedental Entire Function : A non-constant, non-polynomial entire function is called a transcendental entire function.

5. An entire function is analytic at $z=\infty$ iff it is a constant function.
6. An entire function has a pole at $z=\infty$ of order *k* iff it is a polynomial of degree *k*.
7. An entire function has isolated essential singularity at $z=\infty$ iff it is a transcendental entire function.
8. Let $f(z)$ be an entire function then $\lim_{z\to\infty} f(z) = \begin{cases} \text{finite iff } f(z) \text{ is constant} \\ \infty \text{ iff } f(z) \text{ is a polynomial} \\ \text{does not exist iff } f(z) \text{ is transcedental} \end{cases}$

Exercise 4.5

Find the behaviour of the following functions at $z=\infty$

1. $f(z)=\sin z$
2. $f(z)=\sin\frac{1}{z}$
3. $f(z)=\frac{1}{\sin z}$

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4. $f(z)=\dfrac{1}{\sin\left(\dfrac{1}{z}\right)}$ 5. $f(z)=z^2+z+1$ 6. $f(z)=\dfrac{z^2+z+1}{z^3+z^2+z+1}$

7. $f(z)=\dfrac{z^3+1}{z^2+1}$ 8. $f(z)=\dfrac{z^2+4}{z^2+z+7}$ 9. $f(z)=\dfrac{e^z}{e^z+1}$

10. $f(z)=\dfrac{e^z}{e^z-1}$ 11. $f(z)=\dfrac{e^z+1}{e^z}$ 12. $f(z)=\dfrac{e^z-1}{e^z}$

13. $f(z)=\dfrac{e^z-1}{e^z+1}$ 14. $f(z)=\dfrac{e^z+1}{e^z-1}$

S.No.	**Function**	**Behavior in $\mathbb{C}$ i.e. finite complex plane**	**Behavior at $z=\infty$**
1.	$\sin z$	Analytic everywhere	Isolated Essential Singularity
2.	$\sin\left(\dfrac{1}{z}\right)$	Isolated essential singularity at z=0 and analytic elsewhere.	Analytic at $z=\infty$
3.	$\dfrac{1}{\sin z}$	Simple pole at $z=0,\pm\pi,\pm 2\pi,...$ and analytic elsewhere.	Non-Isolated Essential Singularity at $z=\infty$
4.	$\dfrac{1}{\sin\left(\dfrac{1}{z}\right)}$	Simple pole at $z=\pm\dfrac{1}{\pi},\pm\dfrac{1}{2\pi},...$ and non-isolated essential singularity at $z=0$ and analytic elsewhere.	Simple pole at $z=\infty$
5.	$\sin(\pi z)$	Analytic everywhere	Isolated Essential Singularity
6.	$\sin\left(\dfrac{\pi}{z}\right)$	Isolated Essential Singularity at $z=0$ and analytic elsewhere.	Analytic at $z=\infty$
7.	$\dfrac{1}{\sin(\pi z)}$	$0,\pm 1,\pm 2,....$ are simple poles and analytic elsewhere.	Non-Isolated Essential singularity
8.	$\dfrac{1}{\sin\left(\dfrac{\pi}{z}\right)}$	$z=0$ is non-isolated essential singularity and $z=\pm\dfrac{1}{n}$ are simple poles and analytic elsewhere.	$z=\infty$ is simple pole
9.	$\sin(z-a)$	Analytic everywhere	Isolated Essential
10.	$\sin\left(\dfrac{1}{z-a}\right)$	$z=a$, is isolated essential singularity and analytic elsewhere.	Analytic at $z=\infty$
11.	$\dfrac{1}{\sin(z-a)}$	$z=a\pm n\pi$, $n=0,\pm 1,\pm 2,.....$ are simple poles and analytic elsewhere.	Non-Isolated Essential Singularity

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12.	$\frac{1}{\sin\left(\frac{1}{z-a}\right)}$	$z=a$ is non-isolated essential singularity and $z=a\pm\frac{1}{n\pi}$, $n=\pm1,\pm2,.....$ are simple poles and analytic elsewhere.	$z=\infty$ is simple pole
13.	$\sin\pi(z-a)$	Analytic everywhere	Isolated Essential Singularity
14.	$\sin\left(\frac{\pi}{z-a}\right)$	$z=a$, is isolated essential singularity and analytic elsewhere.	Analytic at $z=\infty$
15.	$\frac{1}{\sin\pi(z-a)}$	$z=a\pm n$, $n=0,\pm1,\pm2,.....$ are simple poles and analytic elsewhere.	Non-Isolated Essential
16.	$\frac{1}{\sin\left(\frac{\pi}{z-a}\right)}$	$z=a$ is non-isolated essential singularity, $z=a\pm\frac{1}{n}$ are simple pole and analytic elsewhere.	$z=\infty$ is simple pole
17.	$\cos z$	Analytic everywhere	Isolated Essential
18.	$\cos\frac{1}{z}$	$z=0$, is isolated essential singularity and analytic elsewhere.	$z=\infty$, Analytic
19.	$\frac{1}{\cos z}$	$\pm(2n+1)\frac{\pi}{2}$ is simple pole and analytic elsewhere.	Non-Isolated Essential
20.	$\frac{1}{\cos\left(\frac{1}{z}\right)}$	$z=0$ is non-isolated essential singularity, $z=\frac{1}{\pm(2n+1)\frac{\pi}{2}}$ are simple poles and analytic elsewhere.	analytic at $z=\infty$
21.	e^z	Analytic everywhere	Isolated Essential
22.	$e^{1/z}$	At $z=0$, isolated essential singularity and analytic elsewhere.	Analytic at $z=\infty$
23.	$\frac{1}{e^z}$	Analytic everywhere	Isolated Essential
24.	$\frac{1}{e^{1/z}}$	$z=0$, is an isolated essential singularity and analytic elsewhere.	Analytic at $z=\infty$

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4.6 Residues at a finite point

Residue at a finite point : Let z_0 be an isolated singularity of a function f and R be the positive number s.t. $f(z)$ is analytic at each point z for which $|z-z_0|<R$. The Laurent expansion of $f(z)$ about $z=z_0$ is given by

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}b_n(z-z_0)^{-n} \qquad \ldots\ldots(1)$$

where $a_n=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}}dz$, $b_n=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{-n+1}}dz$

Here C is any circle around z_0 and lying in the domain $|z-z_0|<R$. For $n=1$, the expression for b_n can be written as : $\int_C f(z)\,dz=2\pi i b_1 \quad\Rightarrow\quad b_1=\frac{1}{2\pi i}\int_C f(z)\,dz$

The number b_1, which is the coefficient of $\frac{1}{(z-z_0)}$ in expansion (1) is called the residue of f at the isolated singularity z_0. We use the notation Res($f(z)$; z_0) i.e., $\text{Res}(f(z);z_0)=\frac{1}{2\pi i}\int_C f(z)\,dz$

Theorem 1 : If f is analytic at z_0, then $\text{Res}[f(z);z_0]=0$.

Theorem 2 : If f has a removable singularity at z_0, then we have $\text{Res}[f(z);z_0]=0$.

Theorem 3 : If f has an isolated singularity at z_0 and if f is even in $z-z_0$,

i.e, $f(z-z_0)=f(-(z-z_0))$, then $\text{Res}[f(z);z_0]=0$

Proof : Suppose that f is even in $z-z_0$. Then the Laurent series expansion around z_0 cannot have odd powers of $z-z_0$. Hence, the assertion follows.

Theorem 4 : Residue at a Simple Pole

If f has a simple pole at $z=z_0$, then $\text{Res}(f(z),z_0)=\lim_{z\to z_0}(z-z_0)f(z)$.

Theorem 5 : Residue at a Pole of Order *n*

If f has a pole of order n at $z=z_0$, then $\text{Res}(f(z),z_0)=\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)$

Theorem 6 : Suppose a function f can be written as a quotient $f(z)=\frac{g(z)}{h(z)}$, where g and h are analytic at $z=z_0$. If $g(z_0)\neq 0$ and if the function h has a zero of order 1 at z_0, then f has a simple

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pole at $z=z_0$ and $\text{Res}\big(f(z),z_0\big)=\dfrac{g(z_0)}{h'(z_0)}$

Theorem 7 : Cauchy's Residue Theorem : If *D* is a simple connected domain and *C* is a simple closed contour lying entirely with in *D*. If a function '*f*' is analytic within and on *C*, except at a finite number of isolated singularities $z_1, z_2, ..., z_n$ within *C*, then $\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}\big(f(z),z_k\big)$

Results :

2. If f and g are analytic in a deleted neighborhood of z_0, and if a and $b \in \mathbb{C}$, then

   $\text{Res}\big[a f(z)+b g(z);z_0\big] = a\,\text{Res}\big[f(z);z_0\big]+b\,\text{Res}\big[g(z);z_0\big]$

3. If $f(z)$ has an isolated singularity at $a \in \mathbb{C}$ with nonzero residue at $z=a$, then the residues of $f'(z)$ and $(z-a)f'(z)$ at $z=a$ are zero.

4. If f has an isolated singularity at a, and c is a non-zero complex number, then $f(cz)$ has an isolated singularity at $\dfrac{a}{c}$, and $\text{Res}\left[f(z);\dfrac{a}{c}\right]=\dfrac{1}{c}\text{Res}\big[f(z);a\big]$

8. Let $p(z)$ and $q(z)$ be polynomials with no common zeros, and with degrees m and n, respectively.

   If $f(z)=\dfrac{p(z)}{q(z)}$, then

   (i) f has a removable singularity at ∞, if $n \geq m$

   (ii) $\text{Res}\big[f(z);\infty\big]=0$, if $n \geq m+2$

Exercise 4.6

In problem 1–6, use an appropriate Laurent series to find the indicated residue.

1. $f(z)=\dfrac{2}{(z-1)(z+4)};\text{Res}\big(f(z),1\big)$

2. $f(z)=\dfrac{1}{z^3(1-z)^3};\text{Res}\big(f(z),0\big)$

3. $f(z)=\dfrac{4z-6}{z(2-z)};\text{Res}\big(f(z),0\big)$

4. $f(z)=(z+3)^2\sin\left(\dfrac{2}{z+3}\right);\text{Res}\big(f(z),-3\big)$

5. $f(z)=e^{-\frac{2}{z^2}};\text{Res}\big(f(z),0\big)$

6. $f(z)=\dfrac{e^{-z}}{(z-2)^2};\text{Res}\big(f(z),2\big)$

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In problem 7–16, find the residue at each pole of the given function.

7. $f(z)=\dfrac{z}{z^2+16}$

8. $f(z)=\dfrac{4z+8}{2z-1}$

9. $f(z)=\dfrac{1}{z^4+z^3-2z^2}$

10. $f(z)=\dfrac{1}{\left(z^2-2z+2\right)^2}$

11. $f(z)=\dfrac{5z^2-4z+3}{(z+1)(z+2)(z+3)}$

12. $f(z)=\dfrac{2z-1}{(z-1)^4(z+3)}$

13. $f(z)=\dfrac{\cos z}{z^2(z-\pi)^3}$

14. $f(z)=\dfrac{e^z}{e^z-1}$

15. $f(z)=\sec z$

16. $f(z)=\dfrac{1}{z\sin z}$

In problem 17–20, use Cauchy's residue theorem, where appropriate, to evaluate the given integral along the indicated contours.

17. $\displaystyle\int_C \frac{1}{(z-1)(z+2)^2}dz$ (a) $|z|=\dfrac{1}{2}$ (b) $|z|=\dfrac{3}{2}$ (c) $|z|=3$

18. $\displaystyle\int_C \frac{z+1}{z^2(z-2i)}dz$ (a) $|z|=1$ (b) $|z-2i|=1$ (c) $|z-2i|=4$

19. $\displaystyle\int_C z^3 e^{\frac{-1}{z^2}}\,dz$ (a) $|z|=5$ (b) $|z+i|=2$ (c) $|z-3|=1$

20. $\displaystyle\int_C \frac{1}{z\sin z}dz$ (a) $|z-2i|=1$ (b) $|z-2i|=3$ (c) $|z|=5$

In problem 21–34, use Cauchy's residue theorem to evaluate the given integral along the indicated contour.

21. $\displaystyle\int_C \frac{1}{z^2+4z+13}dz, C:|z-3i|=3$

22. $\displaystyle\int_C \frac{1}{z^3(z-1)^4}dz, C:|z-2|=\frac{3}{2}$

23. $\displaystyle\int_C \frac{z}{z^4-1}dz, C:|z|=2$

24. $\displaystyle\int_C \frac{z}{(z+1)\left(z^2+1\right)}dz, C: 16x^2+y^2=4$

25. $\displaystyle\int_C \frac{ze^z}{z^2-1}dz, C:|z|=2$

26. $\displaystyle\int_C \frac{e^z}{z^3+2z^2}dz, C:|z|=3$

27. $\displaystyle\int_C \frac{\tan z}{z}dz, C:|z-1|=2$

28. $\displaystyle\int_C \frac{\cot \pi z}{z^2}dz, C:|z|=\frac{1}{2}$

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29. $\int_C \cot \pi z \, dz, C$ is the rectangle defined by $x=\frac{1}{2}, x=\pi, y=-1, y=1$

30. $\int_C \frac{2z-1}{z^2\left(z^3+1\right)} dz, C$ is the rectangle defined by $x=-2, x=1, y=-\frac{1}{2}, y=1$

31. $\int_C \left(z^2 e^{\frac{1}{\pi z}} + \frac{ze^z}{z^4-\pi^4} \right) dz, C: 4x^2+y^2=16$

32. $\int_C \frac{\cos z}{(z-1)^2\left(z^2+9\right)} dz, C: |z-1|=1$

33. $\int_C \frac{1}{z^6+1} dz, C$ is the semicircle defined by $y=0, y=\sqrt{4-x^2}$

34. $\int_C e^{\frac{4}{z-2}} dz, C: |z-1|=3$

Answers

1. $\frac{2}{5}$ 2. 6 3. -3 4. $-\frac{4}{3}$ 5. 0 6. $-\frac{1}{e^2}$

7. $\text{Res}\left(f(z),-4i\right)=\frac{1}{2}, \text{Res}\left(f(z),4i\right)=\frac{1}{2}$

8. $\text{Res}\left(f(z),\frac{1}{2}\right)=5$

9. $\text{Res}\left(f(z),1\right)=\frac{1}{3}, \text{Res}\left(f(z),-2\right)=-\frac{1}{12}, \text{Res}\left(f(z),0\right)=-\frac{1}{4}$

10. $\text{Res}\left(f(z),1+i\right)=\frac{-i}{4}, \text{Res}\left(f(z),1-i\right)=\frac{i}{4}$

11. $\text{Res}\left(f(z),-1\right)=6, \text{Res}\left(f(z),-2\right)=-31, \text{Res}\left(f(z),-3\right)=30$

12. $\text{Res}\left(f(z),-3\right)=-\frac{7}{256}$, $\text{Res}\left(f(z),1\right)=\frac{7}{256}$

13. $\text{Res}\left(f(z),0\right)=-\frac{3}{\pi^4}, \text{Res}\left(f(z),\pi\right)=\frac{\left(\pi^2-6\right)}{2\pi^4}$

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14. $\text{Res}\left(f(z), 2n\pi i\right)=1, \quad n=0,\pm1,\pm2,.....$

15. $\text{Res}\left(f(z), \dfrac{(2n+1)\pi}{2}\right)=(-1)^{n+1}, n=0,\pm1,\pm2,...$

16. $\text{Res}\left(f(z),0\right)=0, \text{Res}\left(f(z), n\pi\right)=\dfrac{(-1)^n}{n\pi}, \quad n=\pm1,\pm2,.....$

17. $0;\dfrac{2\pi i}{9};0$ 18. $-\pi+\dfrac{\pi}{2}i;\pi-\dfrac{\pi}{2}i;0$ 19. $\pi i;\pi i;0$ 20. $0;0;0$

21. $\dfrac{\pi}{3}$ 22. $-20\pi i$ 23. 0 24. πi

25. $2\pi i\cosh 1$ 26. $\dfrac{\pi i}{e}\cosh 1$ 27. $-4i$ 28. $-\dfrac{2}{3}\pi^2 i$

29. $6i$ 30. $\pi i+\dfrac{\pi}{\sqrt{3}}$ 31. $\left(\dfrac{1}{3\pi^2}+\dfrac{1}{\pi}\right)i$

32. $-(5\sin 1+\cos 1)\dfrac{\pi i}{25}$ 33. $\dfrac{2\pi}{3}$ 34. $8\pi i$

4.7 Residues at infinity

Residue at the point at infinity : Let f be analytic in a deleted neighborhood of the point at infinity, then $\text{Res}\left(f(z);\infty\right)=-\dfrac{1}{2\pi i}\int_C f(z)dz$.

In other words, $\text{Res}\left(f(z);\infty\right)$ is the negative of the coefficient of $\dfrac{1}{z}$ in the Laurent series expansion of $f(z)$ with centre at the point at infinity.

Results :

1. $\text{Res}\left(f(z);\infty\right)=-\text{Res}\left[\dfrac{1}{z^2}f\left(\dfrac{1}{z}\right);0\right]$
2. If f is analytic at ∞ and has a zero of order $n(\geq 2)$ at ∞, then $\text{Res}\left[f(z);\infty\right]=0$
3. If f is analytic at ∞ and has a simple zero at ∞, then $\text{Res}\left[f(z);\infty\right]=-\lim_{z\to\infty} z\,f(z)$
4. If f has a pole of order n at ∞, then $\text{Res}\left[f(z);\infty\right]=\lim_{z\to\infty}\dfrac{(-1)^n}{(n+1)!}z^{n+2}f^{(n+1)}(z)$

   In the case when f is analytic at ∞, then this formula continues to hold if $n=0$.

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Exercise 4.7

1. Find the residue of $\frac{z^3}{z^2-1}$ at $z=\infty$.

Answers

1. -1

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True-false exercise

1. $\tan z$ has singularities at $z=(2n+1)\dfrac{\pi}{2}$, for $n=0,\pm1,\pm2,....$

2. $z=\dfrac{\pi i}{2}$ is a zero of $\cosh z$.

3. If $p(z)$ is a polynomial in z then the function $f(z)=1/\,p(z)$ can never be an entire function.

4. Suppose a function f has a Taylor series representation with circle of convergence $|z-z_0|=R,\ R>0$. Then f is analytic everywhere on the circle of convergence.

5. Suppose a function f has a Taylor series representation centered at z_0. Then f is analytic everywhere inside the circle of convergence $|z-z_0|=R,\ R>0$, and is not analytic everywhere outside $|z-z_0|=R$.

6. If the function f is entire, then the radius of convergence of a Taylor series expansion of f centered at $z_0=1-i$ is necessarily $R=\infty$.

7. If f is non-constant and analytic at z_0, then $f^{(n)}(z_0)\neq 0$ for some $n\geq 1$.

8. If f is analytic throughout some deleted neighborhood of z_0 and z_0 is a pole of order n, then $\lim\limits_{z\to z_0}(z-z_0)^n f(z)\neq 0$.

9. A singularity of a rational function is either removable or is a pole.

10. The function $f(z)=\dfrac{1}{z^2+2iaz-1}$, $a>1$, has two simple poles within the unit circle $|z|=1$.

11. $z=0$ is a simple pole of $f(z)=-\dfrac{1}{z}+\cot z$.

12. If z_0 is a simple pole of function f, then it is possible that $\operatorname{Res}\big(f(z),z_0\big)=0$.

13. For $a>1$ and b-real, $\displaystyle\int_{|z|=1}\frac{e^{bz}}{a^2z^2+1}dz=\frac{2\pi i\sin(b/a)}{a}$.

14. For $z=x+iy$, $\cosh x$ is never zero and $\cosh z$ has infinitely many zeros when $y\neq 0$.

15. The zeros of $\sin(1/z)$ are $z=1/n\pi(n\in\mathbb{Z})$ and each zero is isolated.

16. If the zeros of an analytic function are not isolated then $f(z)\equiv 0$ throughout the domain of analyticity.

17. An entire function $f(z)$ having $z=\infty$ as a removable singularity is constant.

18. An entire function $f(z)$ has a pole of order n at infinity iff $f(z)$ is a polynomial of degree n.

19. If $f(z)$ has a pole at z_0, then $\exp\{f(z)\}$ has an essential singularity at this point.

20. If $f(z)$ is a nonconstant entire function, then $\exp\{f(z)\}$ has an essential singularity at $z=\infty$.

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21. If a is an isolated singularity of f which is not removable, then a is an essential singularity of $\exp\{f(z)\}$.

22. Poles are isolated. That is, f has a pole of order m at a iff $f(z)=(z-a)^{-m}g(z)$, where g is analytic at a and $g(a)\neq 0$.

23. Suppose f has an essential singularity at $z=a$ and g has a pole at $z=a$. Then the product fg has an essential singularity at $z=a$.

24. The function $f(z)=\dfrac{z^4+1}{(\alpha z-|z|^2)(\beta z-|z|^2)}$, $\alpha,\beta\in\mathbb{C}\setminus\{0\}(\alpha\neq\beta)$ has two simple poles at $z=\overline{\alpha},\overline{\beta}$ and a double pole at $z=0$.

25. The function f defined by the Laurent series $f(z)=\sum_{k=-\infty}^{-1}\dfrac{z^{2k}}{(-k)!}+\sum_{k=0}^{\infty}\dfrac{z^k}{2^{k+1}}$ has an essential singularity at the origin.

26. If f is meromorphic, then f and f' have the same poles and the order of the poles of f' increases by one.

27. The function $f(z)=\dfrac{1}{\sin(1/z)}$ has a singularity at the origin and the Laurent series expansion about the origin does not exist.

28. If f is a rational function such that the degree of its denominator exceeds that of numerator by at least two, then the sum of residues at all the poles is zero.

29. If $f(z)$ is analytic at ∞, then $f'(\infty)=0$.

30. The function $f(z)=\dfrac{e^{1/(z-1)}}{e^z-1}$ has a simple pole at $z=2k\pi i(k\in\mathbb{Z})$ and an essential singularity at $z=1$.

31. If $f(z)=\dfrac{\sin\sqrt{z}}{\sqrt{z}}$, then $z=0$ is a removable singularity of f.

32. If f and g are analytic in a deleted neighborhood of z_0 and if a and $b\in\mathbb{C}$, then $\operatorname{Res}[af(z)+bg(z);z_0]$
$=a\operatorname{Res}[f(z);z_0]+b\operatorname{Res}[g(z);z_0]$

33. If $f(z)$ has an isolated singularity at $a\in\mathbb{C}$ with nonzero residue at $z=a$, then the residues of $f'(z)$ and $(z-a)f'(z)$ at $z=a$ are zero.

34. If f has an isolated singularity at a, and c is a non-zero complex number, then $f(cz)$ has an isolated singularity at a/c, and $\operatorname{Res}[f(cz);a/c]=(1/c)\operatorname{Res}[f(z);a]$.

35. If f is analytic at ∞ and has a zero of order $n(\geq 2)$ at ∞, then $\operatorname{Res}[f(z);\infty]=0$.

36. If f is analytic at ∞ and has a simple zero at ∞, then $\operatorname{Res}[f(z);\infty]=-\lim_{z\to\infty}zf(z)$.

37. There exists an analytic function in the unit disk D such that $f\left(\dfrac{1}{2n}\right)=f\left(\dfrac{1}{2n+1}\right)=\dfrac{1}{n}$ for $n\geq 2$.

38. There exists an analytic function f in the unit disk D such that $f(1/n)=(n+1)/(n-1)$ for $n\geq 2$

39. There exists an analytic function f in the unit disk D such that $f(i^n/n)=-n^{-2}$ for $n\geq 2$.

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Assignment

---------- S C Q ----------

1. The value of $\int_C \frac{z\,dz}{\sin z}$, where $C:|z|=4$ is
   1. $2\pi i$ 2. 0
   3. $-2\pi i$ 4. $4\pi i$

2. The value of $\int_C \tanh z dz$, where $C:|z|=3$ is
   1. 0 2. πi
   3. $2\pi i$ 4. $4\pi i$

3. The value of $\oint_C \frac{(3z+4)}{z(2z+1)}dz$, where C is the circle $|z|=1$ is
   1. $2\pi i$ 2. $3\pi i$
   3. 4 4. -4

4. $\int_C \frac{f(z)}{z-a}dz$ is equal to
   1. $2\pi i f(a)$ 2. $2\pi i \operatorname{Im} f(a)$
   3. $2\pi i$ res $f(a)$ 4. $-2\pi i$ res $f(a)$

5. $\int_C \frac{\tan z}{z^2-1}dz$ has the value around the positive oriented circle $C:|z|=\frac{3}{2}$
   1. $\tan 1$ 2. $2\pi i \tan 1$
   3. $\frac{\tan 1}{2\pi i}$ 4. None of these

6. Value of $\frac{2!}{2\pi i}\int_{|z|=3} \frac{z^2+3z+4}{(z-1)^3}dz$ is
   1. 2 2. 0
   3. πi 4. None of these

7. Let $I=\int_C \frac{\cot(\pi z)}{(z-i)^2}dz$, where C is the contour $4x^2+y^2=2$ (counter clockwise). Then, I is equal to
   1. 0 2. $-2\pi i$
   3. $2\pi i\left(\frac{\pi}{\sinh^2\pi}-\frac{1}{\pi}\right)$ 4. $-\frac{2\pi^2 i}{\sinh^2\pi}$

8. The value of $\int_C \frac{a^n e^{az}}{z^{n+1}n!}dz$, where C is a closed contour around the origin is
   1. $\frac{2\pi i a^{2n}}{n!}$ 2. $\frac{2\pi i a^n}{n!}$
   3. $2\pi i\left(\frac{a^n}{n!}\right)^2$ 4. $\frac{\pi i}{2n!}a^2 n$

9. The value of the integral $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-4)(z-2)}dz$, where C is the circle $|z|=3$ traced anti clockwise, is
   1. $-2\pi i$ 2. $i\pi$
   3. $-i\pi$ 4. $2i\pi$

10. Let Γ be a simple closed curve in the complex plane, then the set of all possible values of $\int_\Gamma \frac{dz}{z(1-z^2)}$ is
    1. $\{0,\pm\pi i\}$ 2. $\{0,\pm\pi i, 2\pi i\}$
    3. $\{0,\pm\pi i, \pm 2\pi i\}$ 4. $\{0\}$

11. If $I=\int_{|z|=2.5} z^2\cot \pi z dz$, then I is equal to
    1. $20\pi i$ 2. $20 i$
    3. $20\pi^2 i$ 4. $10 i$

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12. Let $f(z)=\dfrac{e^z}{(z-1)(z+3)^2}$ and Γ be the circle $|z|=\dfrac{3}{2}$ described in the positive sense. Then $\int_\Gamma f(z)dz,$ has the value

1. 0
2. $\dfrac{\pi ei}{8}$
3. $-\dfrac{\pi ei}{8}$
4. $\dfrac{\pi i\left(e-5e^{-3}\right)}{8}$

13. The value of $\oint_C \dfrac{(4-3z)}{z^2-z}dz,$ where C is a simple closed path (anti clockwise) enclosing the points $z=0$ and $z=1$ is

1. -6π
2. $-6\pi i$
3. $6\pi i$
4. None of these

14. The value of the integral $I=\int_C \dfrac{e^z}{e^z-1}dz$ is where $C:|z|=3\pi$

1. $2\pi i$
2. $3\pi i$
3. $6\pi i$
4. 0

15. Given that 'a' lies inside the closed contour C, the value of the integral $\dfrac{1}{2\pi i}\int_C \dfrac{ze^z}{(z-a)^3}dz$ is

1. $e^a(1+a)$
2. $e^a\left(1+\dfrac{a}{2}\right)$
3. e^a
4. $e^a\left(\dfrac{1+a}{2}\right)$

16. Let $f(z)=\sum_{n=0}^{15} z^n$ for $z\in\mathbb{C}$. If $C:|z-i|=2,$ then $\oint_C \dfrac{f(z)dz}{(z-i)^{15}}$ is equal to

1. $2\pi i(1+15i)$
2. $2\pi i(1-15i)$
3. $2\pi i(1+4i)$
4. $2\pi i$

(GATE 2009)

17. Let T be any circle enclosing the origin and oriented counter clockwise. Then the value of the integral $\int \dfrac{\cos z}{z^2}dz$ is

1. $2\pi i$
2. 0
3. $-2\pi i$
4. undefined

(GATE 2002)

18. Let $\int_C\left[\dfrac{1}{(z-2)^4}-\dfrac{(a-2)^2}{z}+4\right]dz=4\pi,$ where the close curve C is the triangle having vertices at $i,\dfrac{-1-i}{\sqrt{2}}$ and $\dfrac{1-i}{\sqrt{2}}$. The integral being taken in anti clockwise direction. Then, one value of a is

1. $1+i$
2. $2+i$
3. $3+i$
4. $4+i$

(GATE 2012)

19. Let $I=\int_C \dfrac{f(z)}{(z-1)(z-2)}$ where $f(z)=\sin\dfrac{\pi z}{2}+\cos\dfrac{\pi z}{2}$ and C is the curve $|z|=3$ oriented anti clockwise. Then, the value of I is

1. $4\pi i$
2. 0
3. $-2\pi i$
4. $-4\pi i$

(GATE 2010)

20. Let $I=\int_C \dfrac{\cot(\pi z)}{(z-i)^2}dz,$ where C is the contour $4x^2+y^2=2$ (counter clockwise). Then, I is equal to

1. 0
2. $-2\pi i$
3. $2\pi i\left(\dfrac{\pi}{\sinh^2\pi}-\dfrac{1}{\pi}\right)$
4. $-\dfrac{2\pi^2 i}{\sinh^2\pi}$

(GATE 2003)

21. $\int_0^{2\pi}\dfrac{d\theta}{13-5\sin\theta}$ is equal to

1. $-\dfrac{\pi}{6}$
2. $-\dfrac{\pi}{12}$
3. $\dfrac{\pi}{12}$
4. $\dfrac{\pi}{6}$

(GATE 2004)

22. For the positively oriented unit circles, $\oint_{|z|=1}\dfrac{2\,\mathrm{Re}(z)}{z+2}dz$

1. 0
2. πi
3. $2\pi i$
4. $4\pi i$

(GATE 2004)

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23. The value of the integral $\oint_C \frac{dz}{z^2-1}$, $C:|z|=4$ is equal to

1. πi 2. 0
3. $-\pi i$ 4. $2\pi i$

(GATE 2000)

24. $\int_{|z+1|=2} \frac{z^2}{4-z^2}dz$

1. 0 2. $-2\pi i$
3. $2\pi i$ 4. 1

(CSIR NET June 2015)

25. Let $I_r=\int_{C_r} \frac{dz}{z(z-1)(z-2)}$, where $C_r=\{z\in\mathbb{C}:|z|=r\}, r>0$. Then

1. $I_r=2\pi i$ if $r\in(2,3)$
2. $I_r=\frac{1}{2}$ if $r\in(0,1)$
3. $I_r=-2\pi i$ if $r\in(1,2)$
4. $I_r=0$ if $r>3$

(CSIR NET Dec 2011)

26. The function $f(z)=\cos z$ is

1. analytic anywhere
2. singularities at $z=\pm\frac{(n+1)\pi}{2}$
3. singularities at $z=\frac{n\pi}{2}$
4. None of the above

27. The function $f(z)=\frac{1}{z^n}$ is ; $n>0$

1. analytic for all z
2. singularities at $z=0$
3. singularities at $n=0$
4. None of these

28. The poles of function $f(z)=\frac{\sin z}{\cos z}$ are at

1. $\frac{(2n+1)\pi}{2}$, n is any integer
2. $\frac{2n\pi}{3}$, n is any integer
3. $n\pi$
4. None of the above

29. If $z=a$ is an isolated singularity of f, then a is the pole of f, if

1. $\lim_{z\to a}|f(z)|=0$ 2. $\lim_{z\to a}|f(z)|=a$
3. $\lim_{z\to a}|f(z)|=\infty$ 4. None of these

30. If $z=a$ is an isolated singularity of f and $f(z)=\sum_{-\infty}^{\infty}a_n(z-a)^n$ is its Laurent Expansion in $ann(a;0,R)$. Then, $z=a$ is a removable singularity, if

1. $a_n=0, n\le -1$ 2. $a_n\neq 0, n\le -1$
3. $a_n=0, n\ge -1$ 4. $a_n\neq 0, n\ge -1$

31. The simple pole of the function $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ is

1. at $z=-2$ 2. at $z=-1$
3. at $z=+1$ 4. at $z=+2$

32. If $f(z)=\frac{1-e^z}{1+e^z}$, then at $z=\infty$, $f(z)$ have

1. pole
2. removable singularity
3. isolated singularity
4. non isolated singularity

33. If $z=a$ is an isolated singularity of f and $f(z)=\sum_{-\infty}^{\infty}a_n(z-a)^n$ is its Laurent expansion in $ann(a;0,R)$. Also, if $a_{-m}\neq 0$ and $a_n=0$ for $n\le(m+1)$, then

1. $z=a$ is removable singularity
2. $z=a$ is a pole of order m
3. $z=a$ is an essential singularity
4. None of the above

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34. $f(z)=\dfrac{\sin z}{(z-\pi)^2}$ have the pole of order

1. 1 2. 2
3. 3 4. 0

35. If f have an isolated singularity at $z=a$ and $f(z)=\sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion about $z=a$. Then, residue of f at $z=a$ is

1. a_{-1} 2. a_0
3. a_{-2} 4. a_1

36. If $z=a$ is an isolated singularity of f and $f(z)=\sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in $ann(a; 0, R)$. Then, $z=a$ is an essential singularity, if

1. $a_n \neq 0$ for all integers n
2. $a_n = 0$ for all integers n
3. $a_n \neq 0$ for infinitely many negative n
4. $a_n \neq 0$ for many positive n

37. If $z=a$ is an isolated singularity of f and $f(z)=\sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in $ann(a; 0, R)$. Then, if $a_n=0$ for $n \leq -1$, $z=a$ is

1. a pole
2. an essential singularity
3. a removable singularity
4. None of the above

38. The sum of the residues of $f(z)=\dfrac{\sin z}{z\cos z}$ at its poles inside the circle $|z|=2$, is

1. 0 2. 1
3. -1 4. π

39. If f has a pole of order m at $z=a$ and $g(z)=(z-a)^m f(z)$, then

1. $\text{res}(f;a)=\dfrac{1}{(m-1)!} g^{(m-1)}(a)$
2. $\text{res}(f;a)=g^{(m-1)}(a)$
3. $\text{res}(f;a)=\dfrac{1}{(m-1)!} g(a)$
4. None of the above

40. If $z=a$ is an isolated singularity of f and $f(z)=\sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in $ann(a; 0, R)$. Then, $z=a$ is a pole of order m, then

1. $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$
2. $a_{-m} = 0$ and $a_n \neq 0$ for $n \leq -(m+1)$
3. $a_{-m} = 0$ and $a_n = 0$ for $m \leq -(m+1)$
4. None of the above

41. Taylor's expansion of $f(z)=\dfrac{1}{(z+1)^2}$ about the point $z=-i$ is

1. $(2i-2)+\dfrac{1}{z}+\dfrac{1}{z+1}$
2. $\dfrac{i}{2}\left[1+\sum_{n=1}^{\infty}(-1)^n \dfrac{(n+1)(z+i)^n}{(1-i)^n}\right]$
3. $\dfrac{i}{2}\left[1-\sum_{n=1}^{\infty}(-1)^n \dfrac{(n-1)(z+i)^n}{(1-i)^n}\right]$
4. None of the above

42. If $z=a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood, then $z=a$ is called as

1. a removable singularity
2. an isolated singularity
3. an essential singularity
4. None of the above

43. If $z=a$ is an isolated singualarity of f and $f(z)=\sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in $ann(a; 0, R)$. Also, if $a_n \neq 0$ for infinitely many negative integer n, then

1. $z=a$ is a removable singularity
2. $z=a$ is a pole of order m
3. $z=a$ is an essential singularity
4. None of the above

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44. If function $f(z)$ has an isolated singularities at $z=a$, then $z=a$ has removable singularity, if
    1. $\lim_{z\to a}(z-a)=0$ 2. $\lim_{z\to a} f(z)=0$
    3. $\lim_{z\to a}(z-a)f(z)=0$ 4. None of these

45. Which of the following functions does represent the series $\sum_{n=0}^{\infty}\frac{z^n}{n!}$ for $|z|<\infty$?
    1. $\sin z$ 2. $\cos z$
    3. e^z 4. $\log(1+z)$

46. When $0<|z|<4$, the expansion of $\frac{1}{4z-z^2}$ is
    1. $\sum_{n=0}^{\infty}\frac{z^{n+1}}{4^{n+1}}$ 2. $\sum_{n=0}^{\infty}\frac{(-1)^n z^{n+1}}{4^{n+1}}$
    3. $\sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}}$ 4. None of these

47. A function which has poles as its only singularities in the finite part of the plane is said to be
    1. an analytic function
    2. an entire function
    3. a meromorphic function
    4. None of the above

48. For the function $f(z)=e^z$, $z=\infty$ is
    1. isolated essential singularity
    2. pole
    3. ordinary point
    4. None of the above

49. Number of poles of the function $f(z)=\tan\frac{1}{z}$ is
    1. 2 2. 4
    3. infinite 4. None of these

50. The number of isolated singular points of $f(z)=\frac{z+3}{z^2(z^2+2)}$ is
    1. 1 2. 2 3. 3 4. 4

51. If a function $f(z)$ is analytic within a circle *C* with its centre $z=a$ and radius *R*, then for every point *z* inside *C*, $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$, where a_n is equal to
    1. $\frac{f^{(n)}(a)}{n}$ 2. $\frac{f^{(n)}(a)}{n!}$
    3. $f^{(n)}(a)$ 4. None of these

52. The function $f(z)=\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}$ have the poles
    1. $m=1,2$ 2. $m=-1,-2$
    3. $m=2$ 4. None of these

53. The residue of the function $f(z)=\frac{z^2}{(z-1)^2(z-2)}$ is
    1. res $f(-2)=\frac{9}{4}$ 2. res $f(-2)=\frac{3}{2}$
    3. res $f(-2)=\frac{4}{9}$ 4. res $f(-2)=\frac{2}{3}$

54. For the function $f(z)=\frac{z-\sin z}{z^3}$, the point $z=0$ is
    1. a pole of order 3
    2. a pole of order 2
    3. an essential singularity
    4. a removable singularity

55. If $f(z)=z^3$, then it
    1. has an essential singularity at $z=\infty$
    2. has a pole of order 3 at $z=\infty$
    3. has a pole of order 3 at $z=0$
    4. is analytic at $z=\infty$

56. For the function $f(z)=\frac{1-e^{-z}}{z}$, the point $z=0$ is
    1. an essential singularity
    2. a pole of order 0
    3. a pole of order one
    4. a removable singularity

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57. The residue at $z=\infty$ of
$f(z)=\dfrac{z^3}{(z-2)(z-3)(z-5)}$ is
1. 0 2. –10
3. 10 4. None of these

58. The function $f(z)=\dfrac{e^{\tan z}}{z^2+1}$ has
1. countable number of essential singularities
2. only singularity of $f(z)$ are simple poles
3. uncountable number of essential singularity
4. None of the above

59. If $f(z)=\dfrac{1}{e^z-1}$, then
1. f is an entire function
2. f has removable singularities at $z=2n\pi i,\ n\in Z$
3. f has poles at $z=2n\pi i$ with residue 1 for $n\in Z$
4. f has an essential singularity at $z=0$

60. Residue of $f(z)=\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at $z=3$ is
1. $\dfrac{101}{16}$ 2. –8 3. $\dfrac{27}{16}$ 4. 0

61. Residue of $\dfrac{z^3}{z^2-1}$ at $z=\infty$ is
1. 1 2. –1 3. 0 4. ∞

62. A point '*a*' is called an isolated singularity of the function $f(z)$ if in a deleted neighbourhood of *a* the function $f(z)$ is analytic and at *a* the function $f(z)$ is
1. analytic 2. not analytic
3. may be analytic 4. None of these

63. For the function $f(z)=\dfrac{1}{z}\sum_{k=0}^{n}(z+a)^{-k}$, the residue at $z=-a$ when $a\neq 0$ and $a\neq 1$
1. 1 2. $-n$
3. $\dfrac{1-a^n}{(1-a)}$ 4. $-\dfrac{1-a^n}{a^n(1-a)}$

64. For the function $f(z)=\sin\left(\dfrac{x}{|z|^2}-i\dfrac{y}{|z|^2}\right)$ choose the correct answer.
1. $f(z)$ has no singularity
2. $f(z)$ has finite number of singularity with exactly one
3. all the singularities of $f(z)$ are pole
4. infinity is simple pole

65. If the function $f(z)$ is analytic inside and on a simple closed curve *C* and if *a* is a point within *C* and if *C* be traversed in the positive sense, then $f(a)=$
1. $\dfrac{1}{2\pi i}\oint_C f(z)\,dz$
2. $\dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{z-a}dz$
3. $\oint_C \dfrac{f(z)}{z-a}dz$
4. None of these

66. The function $f(z)=z^m e^z$ at $z=\infty$ has
1. non isolated essential singularity
2. pole of order *m*
3. pole of order $m+1$
4. isolated essential singularity

67. The residue of $\dfrac{\sin z}{z^8}$ at $z=0$ is
1. 0 2. $-\dfrac{1}{7!}$
3. $\dfrac{1}{7!}$ 4. None of these

68. Laurent's expansion of the function $\dfrac{1}{(z^2-3z+2)}$ for $|z|>2$ is
1. $\sum_{n=0}^{\infty}\dfrac{2^n-1}{z^{n+1}}$ 2. $\sum_{n=0}^{\infty}\dfrac{2^n}{z^{n+1}}$
3. $\sum_{n=0}^{\infty}\dfrac{2^n+1}{z^{n+1}}$ 4. $\sum_{n=0}^{\infty}\dfrac{2^n}{z^n}$

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69. The residue of $f(z)=\pi \operatorname{cosec} \pi z$ at $z=n$ is

1. $(-1)^n n$ 2. -1 3. 1 4. $(-1)^n$

70. Which of the following does give the residue at $z=\infty$ of any function $f(z)$ is/are

1. $-\text{Res}(z=0)=\frac{1}{z^2} f\left(\frac{1}{z}\right)$
2. $-\frac{1}{2\pi i}\int_C f(z)\,dz$
3. negative of the coefficient of $\frac{1}{z}$ in the expansion of $zf(z)$ in *nbd* of zero
4. All of the above

71. Define a complex function $f(z)$ by

$$f(z)=\begin{cases} e^{\frac{-1}{z^2}} & , \quad z\neq 0 \\ 0 & , \quad z=0 \end{cases} \text{ then}$$

1. at $z=0$, $f(z)$ is continuous
2. at $z=0$, $f(z)$ has removable singularity
3. at $z=0$, $f(z)$ has essential singularity
4. $f(z)$ is analytic at $z=0$

72. The analytical part of Laurent's series is

1. $\sum_{n=1}^{\infty}\frac{a_{-n}}{(z-a)^n}$ 2. $\sum_{n=0}^{\infty} a_n (z-a)^n$
3. $\sum_{n=0}^{\infty} a_n (z-a)^{2n}$ 4. None of these

73. Function g is defined by $g(z)=z^4 \sin\left(\frac{1}{z+1}\right)$ has at infinity

1. a pole of order 4
2. a pole of order 3
3. removable singularity
4. essential singularity

74. The residue of $f(z)$ at $z=2$ where $f(z)=\frac{e^{-z}}{(z-2)^4}$

1. $\frac{1}{6}$ 2. $\frac{e^2}{6}$ 3. $-\frac{1}{6e^2}$ 4. $\frac{1}{6e^2}$

75. The third term in the expansion of $\tan z$ with the help of Maclaurin's series is

1. $\frac{1}{5}z^5$ 2. $\frac{1}{5!}z^5$
3. $\frac{2}{15}z^5$ 4. None of these

76. If $f(z)=\frac{1}{\sinh z}$, then the number of singularities of $f(z)$ on the real line segment $(-n\pi, n\pi)$

1. n 2. $2n-1$
3. $2n+1$ 4. 1

77. The Γ be any circle enclosing the origin and oriented counter clockwise. Then, the value of the integral $\int_\Gamma \frac{\cos z}{z^2} dz$ is

1. 2π 2. 0
3. $-2\pi i$ 4. undefined

78. The general term in the expansion of $\cosh z$ with the help of Taylor's series is

1. $\frac{z^{2n+1}}{(2n+1)!}$ 2. $\frac{z^{2n}}{(2n)!}$
3. $\frac{z^n}{n!}$ 4. None of these

79. If $g(z)$ is analytic at $z=a$ and $f(z)$ has a pole of order k at $z=a$, then $\text{res}_{z=a}\frac{g(z)f'(z)}{f(z)}$ is

1. $kg(a)$ 2. $\frac{-1}{k'}(a)$

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3. $-kg(a)$ 4. $-k!g(a)$

80. If $f(z)$ is analytic and has a pole at $z=a$, then $\lim_{z\to a} f(z)=$
1. 0 2. any constant
3. ∞ 4. None of these

81. Let a be an isolated singularity of $f(z)$ and if $|f(z)|$ is bounded on some neighbourhood of a, then a is
1. removable singularity
2. essential singularity
3. isolated singularity
4. non isolated essential singularity

82. Polynomial $f(z)$ of degree n has a pole of order n at
1. zero 2. infinity
3. $|z|=1$ 4. nowhere

83. The residue of the function $f(z)=\dfrac{2z}{(z+4)(z-1)^2}$ at the point $z=1$ is
1. $\dfrac{1}{5}$ 2. $\dfrac{2}{5}$ 3. $\dfrac{8}{25}$ 4. $\dfrac{4}{25}$

84. If $\sin z=\sum_{n=0}^{\infty} a_n\left(z-\dfrac{\pi}{4}\right)^n$, then a_6 is equal to
1. 0 2. $\dfrac{1}{720}$
3. $\dfrac{1}{720\sqrt{2}}$ 4. $\dfrac{-1}{720\sqrt{2}}$

85. For the function $f(z)=\dfrac{\log_e(z-2)}{(z^2+2z+2)^4}, z=-1-i$ is a pole of order
1. 1 2. 2 3. 3 4. 4

86. If $f(z)$ has a pole at $z=a$, then as $z\to a$
1. $|f(z)|\to 0$ 2. $|f(z)|\to a$
3. $|f(z)|\to \infty$ 4. $|f(z)|\to -\infty$

87. The residue at $z=1$ for $f(z)=\dfrac{\cos z}{z-1}$
1. e 2. e^i
3. $\dfrac{e^i+e^{-i}}{2}$ 4. $\dfrac{e^i-e^{-i}}{2i}$

88. If $f(z)$ has a zero of order n at $z=a$, then $\dfrac{1}{[f(z)]}$ has a pole of order n at $z=$
1. $\dfrac{1}{a}$ 2. 0
3. a 4. None of these

89. The function $\dfrac{1}{z(z-1)^3}$ has a pole of order p and residue r at $z=1$, then
1. $p=1, r=1$ 2. $p=3, r=\dfrac{1}{3}$
3. $p=3, r=1$ 4. $p=1, r=1$

90. If a single valued function $f(z)$ is not defined at $z=a$ but $\lim_{z\to a} f(z)$ exists, then $z=a$ is known as
1. an isolated singularity
2. an essential singularity
3. a removable singularity
4. None of the above

91. Residue of $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ at double pole as $z=-1$ is
1. $\dfrac{4}{25}$ 2. $-\dfrac{4}{5}$ 3. $-\dfrac{14}{25}$ 4. $\dfrac{14}{25}$

92. For the function $f(z)=\dfrac{\pi\cos\pi z}{z^2\sin\pi z}$ which of the following is true
1. $n=1,2,3,\ldots$ are the only simple poles
2. $z=0$ is pole of order 2
3. $z=0$ is pole of order 3
4. residue at $z=0$ is $\dfrac{\pi^2}{3}$

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93. Function $f(z)=\frac{1}{(z+1)(z+3)}$ in the domain $|z|>3$ has a Laurent's expansion given by

1. $\frac{1}{z^2}-\frac{2}{z^3}+\frac{13}{z^4}-\frac{20}{z^5}+\ldots$
2. $\frac{1}{z^2}-\frac{8}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+\ldots$
3. $\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+\ldots$
4. $\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{20}{z^5}+\ldots$

94. For the function $f(z)=e^{\frac{1}{z}}$, $z=0$ is

1. an essential singularity
2. a pole
3. a removable singularity
4. None of the above

95. The expansion of $f(z)=\frac{1}{z(z^2-3z+2)}$ for the region $1<|z|<2$

1. $\frac{1}{2z}+\sum_0^\infty z^n-\frac{1}{4}\sum_0^\infty\left(\frac{z}{2}\right)^n$
2. $\frac{1}{2z}+\sum_0^\infty z^n+\frac{1}{4}\sum_0^\infty\left(\frac{z}{2}\right)^n$
3. $\frac{1}{2z}+\frac{1}{z}\sum_0^\infty z^n+\frac{1}{4}\sum_0^\infty\left(\frac{z}{2}\right)^n$
4. $\frac{1}{2z}-\frac{1}{z}\sum_0^\infty z^{-n}-\frac{1}{4}\sum_0^\infty\left(\frac{z}{2}\right)^n$

96. The coefficient of $\frac{1}{z}$ in the Laurent series of $\frac{\sin z}{z^2}$ is

1. 0 2. 2 3. –1 4. 1

97. For the function $f(z)=\frac{\left(\sin\sqrt{z}\right)}{\sqrt{z}}$, $z=0$ is

1. an essential singularity
2. a pole
3. a removable singularity
4. None of the above

98. At $z=\infty$, the function $f(z)=\cos z-\sin z$ has

1. a removable singularity
2. an essential singularity
3. a non essential singularity
4. None of the above

99. The sum of residues of the function $f(z)=\frac{z^3}{\left[(z-1)(z-2)(z-3)\right]}$ at 1, 2, 3 and ∞ is

1. 0 2. –1
3. 1 4. None of these

100. An example of a function with a non isolated essential singularity at $z=2$ is

1. $\tan\frac{1}{z-2}$ 2. $\sin\frac{1}{z-2}$
3. $e^{-(z-2)}$ 4. $\tan\frac{z-2}{z}$

101. If $\phi(z)=\mathrm{Re}(z)+f(z)$, where $f(z)$ is meromorphic. Then

1. $\phi(z)$ is meromorphic
2. $\phi(z)$ and $f(z)$ has same number of singularities
3. $\phi(z)$ is analytic in every closed and bounded region provided it has no poles
4. $\phi(z)$ is nowhere analytic

102. If the function $f(z)$ has a pole of order $m>1$ at the point $z=a$, then $\mathrm{Res}\, f(a)=$

1. $\frac{1}{m!}\lim_{z\to a}\left[\frac{d^m}{dz^m}\left\{(z-a)^m f(z)\right\}\right]$
2. $\frac{1}{(m-1)!}\lim_{z\to a}\left[\frac{d^m}{dz^m}\left\{f(z)\right\}\right]$

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3. $\frac{1}{(m-1)!}\lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}\right]$
4. None of the above

103. If $f(z)$ is analytic in $D-\{a\}$ and bounded in $0<|z-a|<r$, then
1. $f(z)$ has simple pole at $z=a$
2. $f(z)$ is analytic at $z=a$
3. $f(z)$ has removable singularity at $z=a$
4. $f(z)$ is either analytic at $z=a$ or has removable singularity there at $z=a$

104. The residue of the function $f(z)$ at infinity having simple zero at infinity is
1. $\lim_{z\to\infty}\left[-f(z)\right]$
2. $\lim_{z\to\infty}\left[-z\,f(z)\right]$
3. $\lim_{z\to\infty}\left[-z\,f\left(\frac{1}{z}\right)\right]$
4. None of these

105. The residue of $f(z)=\frac{1}{\left(z^2+a^2\right)^2}$ at $z=ai$ is
1. $\frac{1}{8ia^2}$
2. $\frac{1}{4ia^3}$
3. $\frac{1}{2ia^4}$
4. None of these

106. If the radius of convergence of the series $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ is finite, then $f(z)$ has atleast one singularity
1. on the circle of convergence
2. inside the circle of convergence
3. outside the circle of convergence
4. anywhere

107. The residue of $f(z)=\tanh z$ at the singular point is
1. $-i$ 2. 0 3. 1 4. i

108. If the function $f(z)=\frac{F(z)}{G(z)}$ has a simple pole at $z=a$ such that $F(a)\neq 0$, $G(a)=0$ and $G'(a)\neq 0$ then Res $f(a)=$
1. $\frac{F'(a)}{G(a)}$
2. $\frac{F(a)}{G'(a)}$
3. $\frac{F'(a)}{G'(a)}$
4. None of these

109. The residue of $f(z)=\cot z$ at $z=0$ is
1. i
2. $-i$
3. 1
4. None of these

110. Let $f(z)=u(x,y)+iv(x,y)$ be an entire function having Taylor's series expansion as $\sum_{n=0}^{\infty}a_n z^n$, if $f(x)=u(x,0)$ and $f(iy)=iv(0,y)$, then
1. $a_{2n}=0,\ \forall n$
2. $a_0=a_1=a_2=a_3=0, a_4\neq 0$
3. $a_{2n+1}=0,\ \forall n$
4. $a_0\neq 0$, but $a_2=0$

111. Let $f(z)$ be an analytic function with a simple pole at $z=1$ and a double pole at $z=2$ with residues 1 and -2, respectively. Further if $f(0)=0, f(3)=\frac{-3}{4}$ and f is bounded as $z\to\infty$, then $f(z)$ must be
1. $z(z-3)-\frac{1}{4}+\frac{1}{z-1}-\frac{2}{z-1}+\frac{1}{(z-2)^2}$
2. $-\frac{1}{4}+\frac{1}{z-1}-\frac{2}{z-2}+\frac{1}{(z-2)^2}$
3. $\frac{1}{z-1}-\frac{2}{z-1}+\frac{5}{(z-2)^2}$
4. $\frac{15}{4}+\frac{1}{z-1}+\frac{2}{z-2}-\frac{7}{(z-2)^2}$

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112. In the Laurent series expansion of $f(z)=\frac{1}{z-1}-\frac{1}{z-2}$ valid in the region $|z|>2$, the coefficient of $\frac{1}{z^2}$ is

1. -1 2. 0 3. 1 4. 2

113. Let $f:\mathbb{C}\to\mathbb{C}$ be analytic except for a simple pole at $z=0$ and let $g:\mathbb{C}\to\mathbb{C}$ be analytic. Then, the value of $\frac{\underset{z\to 0}{\text{Res}}\{f(z)g(z)\}}{\underset{z\to 0}{\text{Res}}\, f(z)}$ is

1. $g(0)$ 2. $g'(0)$
3. $\lim_{z\to 0} z\,f(z)$ 4. $\lim_{z\to 0} z\,f(z)g(z)$

(GATE 2011)

114. For the function $f(z)=\sin\left(\frac{1}{\cos\left(\frac{1}{z}\right)}\right)$, the point $z=0$ is

1. a removable singularity
2. a pole
3. an essential singularity
4. a non isolated singularity

(GATE 2009)

115. The coefficient of $\frac{1}{z}$ in the expansion of $\log\left(\frac{z}{z-1}\right)$, valid in $|z|>1$, is

1. -1 2. 1 3. $-\frac{1}{2}$ 4. $\frac{1}{2}$

(GATE 2005)

116. If $f(z)=z^3$, then it

1. has an essential singularity at $z=\infty$
2. has a pole of order 3 at $z=\infty$
3. has a pole of order 3 at $z=0$
4. is analytic at $z=\infty$

(GATE 2001)

117. Consider the function $f(z)=\frac{z^2+\alpha z}{(z+1)^2}$

The residue of $f(z)$ at its pole is equal to 1. Then, the value of α is

1. -1 2. 1 3. 2 4. 3

(GATE 2012)

118. Let $f(z)=\frac{z}{8-z^3}$, $z=x+iy$

$\underset{z=2}{\text{Res}}\, f(z)$ is

1. $-\frac{1}{8}$ 2. $\frac{1}{8}$ 3. $-\frac{1}{6}$ 4. $\frac{1}{6}$

(GATE 2011)

119. Let $\sum_{n=-\infty}^{\infty} b_n z^n$ be the Laurent series expansion of the function $\frac{1}{z\sinh z}$, $0<|z|<\pi$. Then, which one of the following is correct ?

1. $b_{-2}=1, b_0=-\frac{1}{6}, b_2=\frac{7}{360}$
2. $b_{-3}=1, b_{-1}=-\frac{1}{6}, b_1=\frac{7}{360}$
3. $b_{-2}=0, b_0=-\frac{1}{6}, b_2=\frac{7}{360}$
4. $b_0=1, b_2=-\frac{1}{6}, b_1=\frac{7}{360}$

(GATE 2010)

120. Let $\sum_{n=-\infty}^{\infty} a_n(z+1)^n$ be the Laurent series expansion of $f(z)=\sin\left(\frac{z}{z+1}\right)$. Then, a_{-2} is equal to

1. 1 2. 0
3. $\cos(1)$ 4. $-\frac{1}{2}\sin(1)$

(GATE 2009)

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121. Consider the function $f(z)=\dfrac{e^{iz}}{z\left(z^2+1\right)}$

The residue of f at the isolated singular point in the upper half plane $\{z=x+iy\in\mathbb{C}: y>0\}$ is

1. $-\dfrac{1}{2e}$
2. $-\dfrac{1}{e}$
3. $\dfrac{e}{2}$
4. 1

(GATE 2009)

122. Let $f(z)=\cos z-\dfrac{\sin z}{z}$ for non zero $z\in\mathbb{C}$ and $f(0)=0$. Then, $f(z)$ has a zero at $z=0$ of order

1. 0
2. 1
3. 2
4. greater than 2

(GATE 2008)

123. Let $f(z)=\cos z-\dfrac{\sin z}{z}$ for non zero $z\in\mathbb{C}$ and $f(0)=0$. Also, let $g(z)=\sinh z$ for $z\in\mathbb{C}$. Then, $\dfrac{g(z)}{zf(z)}$ has a pole at $z=0$ of order

1. 1
2. 2
3. 3
4. greater than 3

(GATE 2008)

124. Let $f(z)=\dfrac{1}{z^2-3z+2}$, Then the coefficient of $\dfrac{1}{z^3}$ in the Laurent series expansion of $f(z)$ for $|z|>2,$ is

1. 1
2. 1
3. 3
4. 5

(GATE 2007)

125. The sum of the residues at all the poles of $f(z)=\dfrac{\cot \pi z}{(z+a)^2},$ where a is a constant, $a\neq n$ and $a\neq(2n+1)\dfrac{\pi}{2},\quad n\in\mathbb{Z}$ is

1. $\dfrac{1}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{(n+a)^2}-\pi\,\text{cosec}^2\pi a$
2. $-\dfrac{1}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{(n+a)^2}+\pi\,\text{cosec}^2\pi a$
3. $-\dfrac{1}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{(n+a)^2}-\pi\,\text{cosec}^2\pi a$
4. $\dfrac{1}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{(n+a)^2}+\pi\,\text{cosec}^2\pi a$

(GATE 2006)

126. In the Laurent series expansion of $f(z)=\dfrac{1}{z-1}-\dfrac{1}{z-2}$ valid in the region $|z|>2,$ the coefficient of $\dfrac{1}{z^2}$ is

1. –1
2. 0
3. 1
4. 2

(GATE 2004)

127. Let $f(z)$ be an analytic function with a simple pole at $z=1$ and a double pole at $z=2$ with residues 1 and -2, respectively. Further, if $f(0)=0$, $f(3)=-\dfrac{3}{4}$ and f is bounded as $z\to\infty$, then $f(z)$ must be

1. $z(z-3)-\dfrac{1}{4}+\dfrac{1}{z-1}-\dfrac{2}{(z-1)^2}+\dfrac{1}{(z-2)^2}$
2. $-\dfrac{1}{4}+\dfrac{1}{z-1}-\dfrac{2}{z-2}+\dfrac{1}{(z-2)^2}$
3. $\dfrac{1}{z-1}-\dfrac{2}{z-2}+\dfrac{5}{(z-2)^2}$
4. $\dfrac{15}{4}+\dfrac{1}{z-1}+\dfrac{2}{z-2}-\dfrac{7}{(z-2)^2}$

(GATE 2003)

128. An example of a function with a non isolated essential singularity at $z=2$ is

1. $\tan\dfrac{1}{z-2}$
2. $\sin\dfrac{1}{z-2}$
3. $e^{(z-2)}$
4. $\tan\dfrac{z-2}{z}$

(GATE 2003)

129. For the function $f(z)=\sin\dfrac{1}{z}, z=0$ is a

1. removable singularity
2. simple pole
3. branch point
4. essential singularity

(GATE 2002)

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130. For the function $f(z)=\frac{1-e^{-z}}{z}$, the point $z=0$ is
1. an essential singularity
2. a pole of order zero
3. a pole of order one
4. a removable singularity

(GATE 2000)

131. Let *f*, *g* be meromorphic functions on $\mathbb{C}$. If *f* has a zero of order *k* at $z=a$ and *g* has a pole of order *m* at $z=0$, then $g(f(z))$ has
1. a zero of order *km* at $z=a$
2. a pole of order *km* at $z=a$
3. a zero of order $|k-m|$ at $z=a$
4. a pole of order $|k-m|$ at $z=a$

(CSIR NET June 2014)

132. Let $p(z)=a_0+a_1z+...+a_nz^n$ and $q(z)=b_1z+b_2z^2+...+b_nz^n$ be complex polynomials. If a_0, b_1 are non zero complex numbers then the residue of $\frac{p(z)}{q(z)}$ at 0 is equal to

1. $\frac{a_0}{b_1}$ 2. $\frac{b_1}{a_1}$ 3. $\frac{a_1}{b_1}$ 4. $\frac{a_0}{a_1}$

(CSIR NET Dec 2014)

133. Let *D* be the open unit disc in $\mathbb{C}$ and $H(D)$ be the collection of all holomorphic functions on it. Let

$$S=\left\{f\in H(D): f\left(\frac{1}{2}\right)=\frac{1}{2}, f\left(\frac{1}{4}\right)=\frac{1}{4},....., f\left(\frac{1}{2n}\right)=\frac{1}{2n},.....\right\}$$

and

$$T=\left\{f\in H(D): f\left(\frac{1}{2}\right)=f\left(\frac{1}{3}\right)=\frac{1}{2}, f\left(\frac{1}{4}\right)=f\left(\frac{1}{5}\right)=\frac{1}{4},...., f\left(\frac{1}{2n}\right)=f\left(\frac{1}{2n+1}\right)=\frac{1}{2n},...\right\}$$

Then
1. Both S, T are singleton sets.
2. *S* is a singleton set but $T=\phi$.
3. *T* is a singleton set but $S=\phi$.
4. Both *S*, *T* are empty.

(CSIR NET June 2016)

134. Number of zeros of the function $f(z)=\sin\left(\frac{1}{z}\right)$ is

1. 3 2. 4
3. infinite 4. None of these

135. The zero of first order is known as

1. complex zero 2. simple zero
3. singularity 4. None of these

136. For second order zero
1. $f(z_0)=f'(z_0)=0$ and $f''(z_0)\neq 0$
2. $f(z_0)=0, f'(z_0)\neq 0$ and $f''(z_0)\neq 0$
3. $f(z_0)=f'(z_0)=f''(z_0)=0$
4. None of the above

137. **(A)** Every analytic function can be represented by power series.
(B) Taylor series is a power series.
1. A and B both false
2. A and B both true
3. A is true and B is false
4. A is false and B is true

138. If an analytic function $f(z)$ has a zero of *n*th order at $z=\infty$, then the function $f\left(\frac{1}{z}\right)$ has such a zero

1. at $z=\infty$ 2. at each point *z*
3. at $z=0$ 4. None of these

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139. If $f(a)=0,\ f'(a)=0,\ f''(a)\neq 0,$ then at $z=a$ the function $f(z)$ is said to have a zero of

1. first order
2. second order
3. third order
4. None of these

140. The function $f(z)=e^z-2i$

1. has no zeros
2. has only one zero
3. has a countable number of zeros
4. has an uncountable number of zeros

141. Let *S* be the disk $|z|<3$ in the complex plane and let $f:S\to\mathbb{C}$ be an analytic function such that $f\left(1+\frac{\sqrt{2}}{n}i\right)=-\frac{2}{n^2}$ for each natural number *n*. Then, $f\left(\sqrt{2}\right)$ is equal to

1. $3-2\sqrt{2}$
2. $3+2\sqrt{2}$
3. $2-3\sqrt{2}$
4. $2+3\sqrt{2}$

(GATE 2008)

142. For every path between the limits $\int_{-2}^{-2+i}(2+z)^2\,dz=$

1. $\frac{i}{2}$
2. $\frac{i}{3}$
3. $\frac{-i}{3}$
4. $\frac{-i}{2}$

143. If $f(z)$ is entire function the Taylor series is

1. convergent for all *z*
2. divergent for all *z*
3. constant
4. None of these

144. Let *C* be the circle $|z|=\frac{3}{2}$ in the complex plane that is oriented in the counter clockwise direction. The value of *a* for which $\int_C\left(\frac{z+1}{z^2-3z+2}+\frac{a}{z-1}\right)dz=0$ is

1. 1
2. –1
3. 2
4. –2

(CSIR NET Dec 2016)

145. Let f be a holomorphic function on $0<|z|<\varepsilon,\ \varepsilon>0$ given by a convergent Laurent series $\sum_{n=-\infty}^{\infty}a_n z^n$.

Given also that $\lim_{z\to 0}|f(z)|=\infty,$

We can conclude that

1. $a_{-1}\neq 0$ and $a_{-n}=0$ for all $n\geq 2$
2. $a_{-N}\neq 0$ for some $N\geq 1$ and $a_{-n}=0$ for all $n>N$
3. $a_{-n}=0$ for all $n\geq 1$
4. $a_{-n}\neq 0$ for all $n\geq 1$

(CSIR NET Dec 2016)

146. Let f be an entire function such that $f(x)=-f(-x)$ for every real *x*. then

1. f is constant function
2. f is even function
3. f is odd function
4. none of these

147. Let f and g are two analytic functions in the domain $D:|z|<2$ such that neither f non g has zero in *D*. If $\frac{f'}{f}\left(\frac{1}{n}\right)=\frac{g'}{g}\left(\frac{1}{n}\right)\ \forall\ n\in\mathbb{N}$ then

1. $f(z)=c\,g(z)$
2. f and g are constant functions
3. $f\equiv 0\equiv g$
4. None of these

148. Let f be an analytic function on open unit disk $D:|z|\leq 1$ such that $\int_{|z|=1}\frac{f(z)}{\left((n+1)z-1\right)^2}dz=0\ \forall\ n\in\mathbb{N}$ then

1. f is constant
2. f is linear polynomial
3. $f\equiv 0$
4. None of these

149. The function $f(z)=|z|^2+i\bar{z}+1$ is differentiable at

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1. i 2. 1
3. $-i$ 4. no point in $\mathbb{C}$

(Gate 2014)

150. Suppose f is holomorphic in an open neighbourhood of $z_0 \in \mathbb{C}$. Given that the series $\sum_{n=0}^{\infty} f^{(n)}(z_0)$ converges absolutely, we can conclude that
1. f is constant
2. f is a polynomial
3. f can be extended to an entire function
4. $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$

(CSIR NET June 2017)

151. Let D be the open unit disc in the complex plane and $U = D \setminus \left\{-\frac{1}{2}, \frac{1}{2}\right\}$. Also, let $H_1 = \{f : D \to \mathbb{C} \mid f$ is holomorphic and bounded $\}$ and $H_2 = \{f : U \to \mathbb{C} \mid f$ is holomorphic and bounded $\}$.
Then the map $r : H_1 \to H_2$ given by $r(f) = f|_U$, the restriction of f to U, is
1. injective but not surjective
2. surjective but not injective
3. injective and surjective
4. neither injective nor surjective

(CSIR NET Dec 2017)

152. The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z + e^{-z}$ has
1. finitely many zeros
2. no zeros
3. only real zeros
4. has infinitely many zeros

(CSIR NET Dec 2017)

153. Let $f : \{z \mid |z| < 1\} \to \mathbb{C}$ be a non-constant analytic function. Which of the following conditions can possibly be satisfied by f?
1. $f\left(\frac{1}{n}\right) = f\left(\frac{-1}{n}\right) = \frac{1}{n^2} \ \forall\ n \in \mathbb{N}$
2. $f\left(\frac{1}{n}\right) = f\left(\frac{-1}{n}\right) = \frac{1}{2n+1} \ \forall\ n \in \mathbb{N}$
3. $\left|f\left(\frac{1}{n}\right)\right| < 2^{-n} \ \forall\ n \in \mathbb{N}$
4. $\frac{1}{\sqrt{n}} < \left|f\left(\frac{1}{n}\right)\right| < \frac{2}{\sqrt{n}} \ \forall\ n \in \mathbb{N}$

(CSIR NET June 2018)

154. The value of the integral $\oint_{|1-z|=1} \frac{e^z}{z^2-1} dz$ is
1. 0 2. $(\pi i)e$
3. $(\pi i)e - (\pi i)e^{-1}$ 4. $\left(e + e^{-1}\right)$

(CSIR NET June 2018)

---------- M C Q ----------

1. At $z = 0$ the function $f(z) = \frac{e^z + 1}{e^z - 1}$ has
   1. a removable singularity.
   2. a pole.
   3. an essential singularity.
   4. the residue of $f(z)$ is 2.

   (CSIR NET June 2011)

2. Let f be an entire function such that $\lim_{|z| \to \infty} |f(z)| = \infty$. Then
   1. $f\left(\frac{1}{z}\right)$ has an essential singularity at 0.
   2. f cannot be a polynomial
   3. f has finitely many zeros

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4. $f\left(\frac{1}{z}\right)$ has a pole at 0

(CSIR NET Dec 2011)

3. At $z=0$, the function $f(z)=\exp\left(\frac{z}{1-\cos z}\right)$ has
   1. a removable singularity.
   2. a pole.
   3. an essential singularity.
   4. the Laurent expansion of $f(z)$ around $z=0$ has infinitely many positive and negative powers of *z*.

(CSIR NET June 2012)

4. Which of the following functions *f* are entire functions and have simple zeros at $z=ik$ for all $k\in\mathbb{Z}$.
   1. $f(z)=a_n z^n+a_{n-1}z^{n-1}+...+a_0$ for some $n\geq 1$ and some $a_0,a_1,...,a_n\in\mathbb{C}$.
   2. $f(z)=a\sin 2\pi i z$, for some $a\in\mathbb{C}$.
   3. $f(z)=b\cos 2\pi\left(iz-\frac{1}{4}\right)$, for some $b\in\mathbb{C}$.
   4. $f(z)=e^{cz}$, for some $c\in\mathbb{C}$.

(CSIR NET Dec 2012)

5. Let $\gamma_k=\left\{ke^{ik\theta}:0\leq\theta\leq 2\pi\right\}$ for $k=1,2,3$. Which of the following are necessarily correct ?
   1. $\frac{1}{2\pi i}\int_{\gamma_k}\frac{1}{z}dz=0$ for $k=1,2,3$.
   2. $\frac{1}{2\pi i}\int_{\gamma_1}\frac{1}{z}dz=1$.
   3. $\frac{1}{2\pi i}\int_{\gamma_2}\frac{1}{z}dz=4$.
   4. $\frac{1}{2\pi i}\int_{\gamma_3}\frac{1}{z}dz=3$.

(CSIR NET Dec 2012)

6. Consider the function $f(z)=z^2(1-\cos z)$, $z\in\mathbb{C}$. Which of the following are correct ?
   1. The function *f* has zeros of order 2 at 0
   2. The function *f* has zeros of order 1 at $2\pi n, n=\pm1,\pm2,...$
   3. The function *f* has zeros of order 4 at 0
   4. The function *f* has zeros of order 2 at $2\pi n$, $n=\pm1,\pm2,...$

(CSIR NET June 2013)

7. Let $f(z)=\frac{z-1}{\exp\left(\frac{2\pi i}{z}\right)-1}$. Then,
   1. *f* has an isolated singularity at $z=0$.
   2. *f* has a removable singularity at $z=1$.
   3. *f* has infinitely many poles.
   4. each pole of *f* is of order 1.

(CSIR NET Dec 2013)

8. For $z\in\mathbb{C}$, define $f(z)=\frac{e^z}{e^z-1}$. Then
   1. *f* is entire.
   2. the only singularities of *f* are poles.
   3. *f* has infinitely many poles on the imaginary axis.
   4. each pole of *f* is simple.

(CSIR NET June 2014)

9. Let $f(z)=\frac{1}{e^z-1}$ for all $z\in\mathbb{C}$ such that $e^z\neq 1$. Then
   1. *f* is meromorphic.
   2. the only singularities of *f* are poles.
   3. *f* has infinitely many poles on the imaginary axis.
   4. Each pole of *f* is simple.

(CSIR NET Dec 2015)

10. Let *f* be a holomorphic function on the unit disc $\{|z|<1\}$ in the complex plane. Which of the following is/are necessarily true ?
    1. If for each positive integer *n* we have $f\left(\frac{1}{n}\right)=\frac{1}{n^2}$ then $f(z)=z^2$ on the unit disc.
    2. If for each positive integer *n* we have $f\left(1-\frac{1}{n}\right)=\left(1-\frac{1}{n}\right)^2$ then $f(z)=z^2$ on the unit disc.
    3. *f* cannot satisfy $f\left(\frac{1}{n}\right)=\frac{(-1)^n}{n}$ for each positive integer *n*.

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4. f cannot satisfy $f\left(\frac{1}{n}\right)=\frac{1}{n+1}$ for each positive integer n.

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11. Let f be an analytic function defined on the open unit disc in $\mathbb{C}$. Then f is constant if
1. $f\left(\frac{1}{n}\right)=0$ for all $n\geq 1$.
2. $f(z)=0$ for all $|z|=\frac{1}{2}$.
3. $f\left(\frac{1}{n^2}\right)=0$ for all $n\geq 1$.
4. $f(z)=0$ for all $z\in(-1,1)$.

(CSIR NET June 2015)

12. Let f be an analytic function in $\mathbb{C}$. Then f is constant if the zero set of f contains the sequence
1. $a_n=\frac{1}{n}$
2. $a_n=(-1)^{n-1}\frac{1}{n}$
3. $a_n=\frac{1}{2n}$
4. $a_n=n$ if 4 does not divide n and $a_n=\frac{1}{n}$ if 4 divides n.

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13. Consider the function $F(z)=\int_1^2\frac{1}{(x-z)^2}dx$, $\mathrm{Im}(z)>0$. Then there is a meromorphic function $G(z)$ on $\mathbb{C}$ that agrees with $F(z)$ when $\mathrm{Im}(z)>0$, such that
1. $1,\infty$ are poles of $G(z)$.
2. $0,1,\infty$ are poles of $G(z)$.
3. $1,2$ are poles of $G(z)$.
4. $1,2$ are simple poles of $G(z)$.

(CSIR NET June 2016)

14. Let $f:\mathbb{C}\to\mathbb{C}$ be a meromorphic function analytic at 0 satisfying $f\left(\frac{1}{n}\right)=\frac{n}{2n+1}$ for $n\geq 1$. Then
1. $f(0)=\frac{1}{2}$
2. f has a simple pole at $z=-2$
3. $f(2)=\frac{1}{4}$
4. no such meromorphic function exists

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15. Let $D=\{z\in\mathbb{C}:|z|<1\}$ be the unit disc. Let $f:D\to\mathbb{C}$ be an analytic function satisfying $f\left(\frac{1}{n}\right)=\frac{2n}{3n+1}$ for $n\geq 1$. Then
1. $f(0)=\frac{2}{3}$
2. f has a simple pole at $z=-3$
3. $f(3)=\frac{1}{3}$
4. no such f exists

(CSIR NET June 2011)

16. Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function and let $g:\mathbb{C}\to\mathbb{C}$ be defined by $g(z)=f(z)-f(z+1)$ for $z\in\mathbb{C}$. Which of the following statements are true ?
1. if $f\left(\frac{1}{n}\right)=0$ for all positive integers n, then f is a constant function
2. if $f(n)=0$ for all positive integers n, then f is a constant function
3. if $f\left(\frac{1}{n}\right)=f\left(\frac{1}{n}+1\right)$ for all positive integers n, then g is a constant function
4. if $f(n)=f(n+1)$ for all positive integers n, then g is a constant function

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17. Let $f(z)$ be the meromorphic function given by $\frac{z}{(1-e^z)\sin z}$. Then

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1. $z=0$ is a pole of order 2.
2. for every $k\in\mathbb{Z}$, $z=2\pi ik$ is a simple pole.
3. for every $k\in\mathbb{Z}\setminus\{0\}, z=k\pi$ is a simple pole.
4. $z=\pi+2\pi i$ is a pole.

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18. Let f be analytic on the open unit disk $D:|z|\le 1$ such that

$$\int_{|z|=1}\frac{f(z)}{(n+1)z-1}dz=0 \quad \forall\ n\in\mathbb{N}$$ then

1. f is constant
2. $f\equiv 0$
3. f is polynomial of degree atmost two
4. None of these

19. Let $f=u+iv$ be an entire function where u,v are the real and imaginary parts of f respectively. If the Jacobian matrix

$$J_a=\begin{bmatrix}u_x(a) & u_y(a)\\ v_x(a) & v_y(a)\end{bmatrix}$$

is symmetric for all $a\in\mathbb{C}$, then

1. f is a polynomial.
2. f is a polynomial of degree ≤ 1.
3. f is necessarily a constant function.
4. f is a polynomial of degree strictly greater than 1.

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20. Consider the function $f(z)=\dfrac{\sin(\pi z/2)}{\sin(\pi z)}$.

Then f has poles at

1. all integers
2. all even integers
3. all odd integers
4. all integers of the form $4k+1,\ k\in\mathbb{Z}$

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21. Let $p(z)=z^n+a_{n-1}z^{n-1}+\ldots+a_0$, where $a_0,\ldots,a_{n-1}$ are complex numbers and let $q(z)=1+a_{n-1}z+\ldots+a_0z^n$. If $|p(z)|\le 1$ for all z with $|z|\le 1$ then

1. $|q(z)|\le 1$ for all z with $|z|\le 1$
2. $q(z)$ is a constant polynomial
3. $p(z)=z^n$ for all complex numbers z
4. $p(z)$ is a constant polynomial

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22. Let f be an entire function. Consider $A=\{z\in\mathbb{C}\mid f^{(n)}(z)=0$ for some positive integer $n\}$. Then

1. if $A=\mathbb{C}$, then f is a polynomial
2. if $A=\mathbb{C}$, then f is a constant function
3. if A is uncountable, then f is a polynomial
4. if A is uncountable, then f is a constant function

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23. Suppose that $f:\mathbb{C}\to\mathbb{C}$ is an analytic function. Then f is a polynomial if

1. for any point $a\in\mathbb{C}$, if $f(z)=\sum_0^\infty a_n(z-a)^n$ is a power series expansion at a, then $a_n=0$ for at least one n
2. $\lim_{|z|\to\infty}|f(z)|=\infty$
3. $\lim_{|z|\to\infty}|f(z)|=M$ for some M
4. $|f(z)|\le M|z|^n$ for $|z|$ sufficiently large and for some n

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24. Let Ω be an open connected subset of $\mathbb{C}$. Let $E=\{z_1,z_2,\ldots,z_r\}\subseteq\Omega$. Suppose that $f:\Omega\to\mathbb{C}$ is a function such that $f_{|(\Omega\setminus E)}$ is analytic. Then f is analytic on Ω if

1. f is continuous on Ω
2. f is bounded on Ω
3. for every j, if $\sum_{m\in\mathbb{Z}}a_m(z-z_j)^m$ is Laurent series expansion of f at z_j,

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then $a_m = 0$ for $m = -1, -2, -3, \ldots.$

4. for every j, if $\sum_{m \in \mathbb{Z}} a_m \left(z - z_j\right)^m$ is

Laurent series expansion of f at z_j,

then $a_{-1} = 0$

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True false key

1. T	2. T	3. F	4. T
5. F	6. T	7. T	8. T
9. T	10. F	11. F	12. F
13. T	14. T	15. T	16. F
17. F	18. T	19. T	20. T
21. T	22. T	23. T	24. F
25. T	26. F	27. F	28. T
29. T	30. T	31. T	32. T
33. F	34. T	35. T	36. T
37. F	38. T	39. F	

Assignment key

S C Q

1. 2	2. 4	3. 2	4. 3
5. 2	6. 1	7. 3	8. 3
9. 3	10. 3	11. 2	12. 2
13. 2	14. 3	15. 2	16. 1
17. 2	18. 3	19. 4	20. 3
21. 4	22. 1	23. 2	24. 3
25. 4	26. 1	27. 2	28. 1
29. 3	30. 1	31. 1	32. 4
33. 2	34. 1	35. 1	36. 3
37. 3	38. 1	39. 1	40. 1
41. 2	42. 2	43. 3	44. 3
45. 3	46. 3	47. 3	48. 1
49. 3	50. 3	51. 2	52. 1
53. 3	54. 4	55. 2	56. 4
57. 2	58. 1	59. 3	60. 3
61. 2	62. 2	63. 4	64. 2
65. 2	66. 4	67. 2	68. 1
69. 4	70. 4	71. 3	72. 2
73. 2	74. 3	75. 3	76. 4
77. 2	78. 2	79. 3	80. 3
81. 1	82. 2	83. 3	84. 4
85. 4	86. 3	87. 3	88. 3
89. 3	90. 3	91. 3	92. 3
93. 3	94. 1	95. 4	96. 4
97. 3	98. 2	99. 1	100. 1
101. 4	102. 3	103. 4	104. 2
105. 2	106. 1	107. 3	108. 2
109. 3	110. 1	111. 2	112. 1
113. 1	114. 4	115. 2	116. 2
117. 4	118. 3	119. 1	120. 4
121. 1	122. 3	123. 2	124. 3
125. 1	126. 1	127. 2	128. 1
129. 4	130. 4	131. 2	132. 1
133. 2	134. 3	135. 2	136. 1
137. 2	138. 3	139. 2	140. 3
141. 1	142. 3	143. 1	144. 3
145. 2	146. 3	147. 1	148. 1
149. 3	150. 3	151. 3	152. 4
153. 1	154. 2		

M C Q

1. 2,4	2. 3,4	3. 3,4	4. 2,3
5. 2,4	6. 3,4	7. 2,3,4	8. 2,3,4
9. 1,2,3,4	10. 1,3	11. 1,2,3,4	12.1,2,3,4
13. 3,4	14. 1,2,3	15. 1,2,3	16. 1,3
17. 2,3	18. 1,2	19. 1,2	20. 3,4
21. 1,2,3	22. 1,3	23. 1,2,3,4	24. 1,3

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Chapter - 5

Some transformations

5.1 Linear Transformations

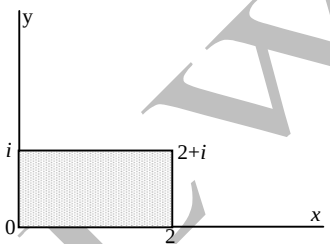
Def. Linear Transformation : A function $w = f(z) = az + b$, where '*a*' and '*b*' are both complex constants, defined on *z*-plane is called complex linear transformation.

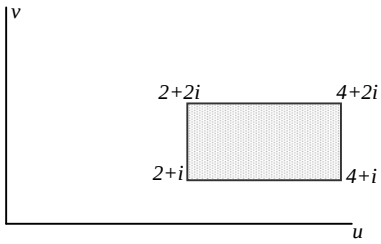
There are three basic linear transformation :

(i) Linear Translation (ii) Rotation (iii) Magnification

Def. Linear Translation: In a Linear Translation $w = az + b$; $a = 1$ and '*b*' is arbitrary complex number i.e., it is of the form $w = z + b$.

eg. Consider the case of given rectangle and linear translation $w = z + 2 + i$ .....(1)





Put $z = 0$ in **(1)** $\Rightarrow w = 0 + 2 + i = 2 + i$

$z = 2$ $\Rightarrow w = 2 + 2 + i = 4 + i$

$z = i$ $\Rightarrow w = i + 2 + i = 2 + 2i$

$z = 2 + i$ $\Rightarrow w = 2 + i + 2 + i = 4 + 2i$

Remark : Translation mapping does not change the shape and size of the figure.

Def. Rotation : The linear transformation of the type $w = az$; where $|a| = 1$ is called a rotation.

OR

$w = e^{i\theta}.z$; where θ is a real number is called rotation.

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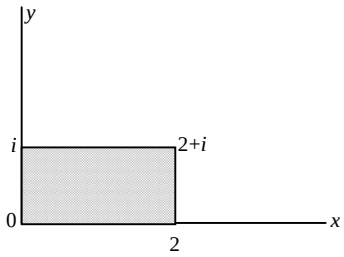
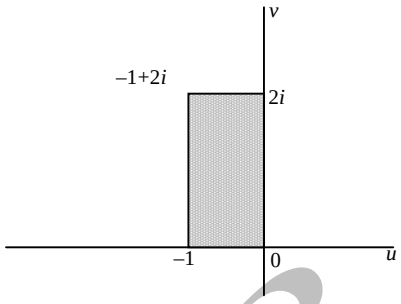
eg. Again consider the case of above rectangle and rotation $w = iz$ we have





Put $z = 0 \quad \Rightarrow w = i0 = 0$

$z = 2 \quad \Rightarrow w = 2i$

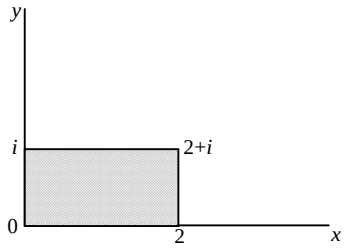
$z = 2 + i \quad \Rightarrow w = -1 + 2i$

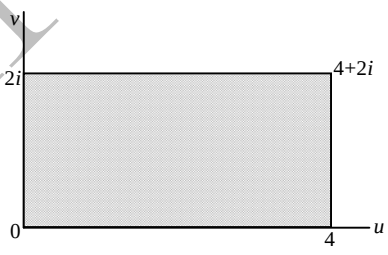
$z = i \quad \Rightarrow w = -1$

Remark : A rotation does not change the size and shape of a figure.

Def. Magnification : A linear transformation of the form $w = az$, where $a > 0$ *i.e* '*a*' is real number greater than zero, then it is called magnification.

e.g. Consider the case of above rectangle and the magnification $w = 2z$ we have





Remark : Magnification mapping does not change the shape of the figure but it changes the size of the figure provided $a \neq 1$.

Exercise 5.1

1. Find the image of the closed disc $|z| \leq 1$ under the given linear mapping $w = f(z)$

(i) $f(z) = z + 3i$ (ii) $f(z) = z + 2 - i$ (iii) $f(z) = 3iz$

(iv) $f(z) = (1+i)z$ (v) $f(z) = 2z - i$ (vi) $f(z) = (6-5i)z + 1 - 3i$

2. Find the image of the triangle with vertices 0,1 and *i* under the given linear mapping $w = f(z)$

(i) $f(z) = z + 2i$ (ii) $f(z) = 3z$ (iii) $f(z) = e^{\frac{i\pi}{4}} z$

(iv) $f(z) = \frac{1}{2} iz$ (v) $f(z) = -3z + i$ (vi) $f(z) = (1-i)z - 2$

3. Express the given linear mapping $w = f(z)$ as a composition of a rotation, magnification and translation

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(i) $f(z)=3iz+4$ (ii) $f(z)=5e^{\frac{i\pi}{5}}z+7i$ (iii) $f(z)=-\frac{1}{2}z+1-\sqrt{3}i$

4. Which one of the following is true?

(a) $ToR=RoT$ (b) $ToM=MoT$ (c) $RoM=MoR$ (d) None of these.

Answers

1. (i) $|w-3i|\le 1$ (ii) $|w-2+i|\le 1$ (iii) $|w|\le 3$

(iv) $|w|\le\sqrt{2}$ (v) $|w+i|\le 2$ (vi) $|w-1+3i|\le\sqrt{61}$

2. (i) Triangle with vertices $2i,1+2i$ and $3i$

(ii) Triangle with vertices $0,3$ and $3i$

(iii) Triangle with vertices $0,\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}i$ and $-\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}i$

(iv) Triangle with vertices $0,\frac{1}{2}i$ and $-\frac{1}{2}$

(v) Triangle with vertices $i,-3+i$ and $-2i$

(vi) Triangle with vertices $-2,-1-i$ and $i-1$

3. (i) $f(z)=T\circ M\circ R(z)$ where $R(z)=e^{\frac{\pi}{2}i}z$, $M(z)=3z$ and $T(z)=z+4$

(ii) $f(z)=T\circ M\circ R(z)$ where $R(z)=e^{\frac{\pi}{5}i}z$, $M(z)=5z$ and $T(z)=z+7i$

(iii) $f(z)=T\circ M\circ R(z)$ where $R(z)=e^{\pi i}z$, $M(z)=\frac{1}{2}z$ and $T(z)=z+1-\sqrt{3}i$ 4. (c)

5.2 Bilinear Transformation

Def. Bilinear Transformation (Mobius transformation) (Linear fractional transformation) : A mapping $T(z)=\frac{az+b}{cz+d}$, where a,b,c,d are complex constants is called bilinear transformation if $ad-bc\neq 0$.

Note : If $ad-bc=0$ then it becomes a constant mapping.

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Results :

1. A bilinear transformation maps circles or straight lines onto circles or straight lines.
2. If pole of the bilinear transformation lies on the boundary then the image is straight line and if it is not on the boundary then image is circle.
3. The set of all bilinear transformation forms a non-abelian group under composition of mappings.

Exercise 5.2

In problem 1–4 find the images of the points 0, 1, i and ∞ under the given linear fractional transformation.

1. $f(z)=\dfrac{i}{z}$ 2. $f(z)=\dfrac{2}{z-i}$ 3. $f(z)=\dfrac{z+i}{z-i}$ 4. $f(z)=\dfrac{z-1}{z}$

In the problems 5-8, find the image of the disks $|z|\le 1$ and $|z-i|\le 1$ under the given linear fractional transformation f.

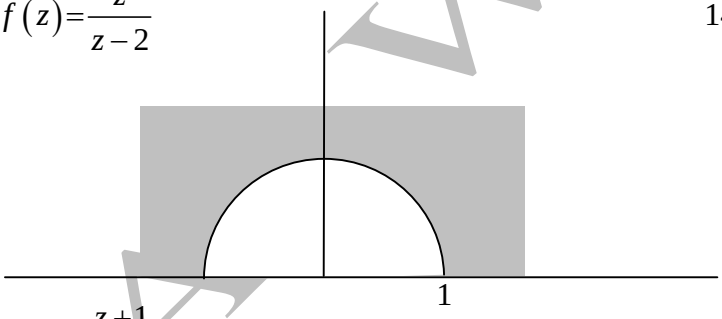
5. $f(z)=\dfrac{i}{z}$ 6. $f(z)=\dfrac{2}{z-i}$ 7. $f(z)=\dfrac{z+i}{z-i}$ 8. $f(z)=\dfrac{z-1}{z}$

In problems 9-12, find the images of half-planes $x\ge 0$ and $y\le 1$ under the given linear fractional transformation :-

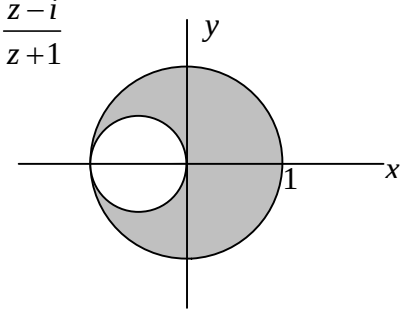
9. $f(z)=\dfrac{i}{z}$ 10. $f(z)=\dfrac{2}{z-i}$ 11. $f(z)=\dfrac{z+i}{z-i}$ 12. $f(z)=\dfrac{z-1}{z}$

In problems 13–16, find the image of the shaded region under the given bilinear transformation.

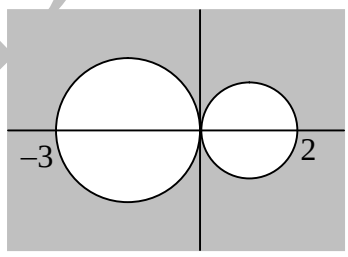
13. $f(z)=\dfrac{z}{z-2}$



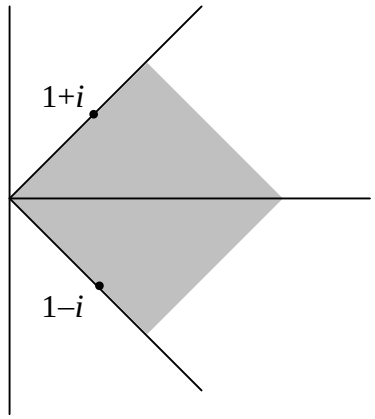
14. $f(z)=\dfrac{z-i}{z+1}$



15. $f(z)=\dfrac{z+1}{z-2}$



16. $f(z)=\dfrac{-z-1+i}{z-1+i}$



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17. In the transformation $z=\dfrac{i-w}{i+w}$, show that positive half of *w*-plane given by $v\geq 0$ correspond to the circle $|z|\leq 1$ in the *z*-plane.

18. Show that the transformation $w=\dfrac{5-4z}{4z-2}$ transforms the circle $|z|=1$ into a circle of radius unity in the *w* plane. Also find the centre of the circle.

19. Show that the relation $w=\dfrac{iz+z}{4z+i}$ transforms the real axis in *z*-plane to a circle in the *w*-plane. Find the centre and radius of the circle and the point in the *z*-plane which is mapped on the centre of the circle.

20. Find the radius and the centre of the circle in the *w*-plane, which corresponds to real axis in *z*-plane where $w=\dfrac{ze^{a}-i}{z-ie^{a}}$, *a* being a real constant.

21. Show that the transformation $\bar{a}wz-bw-\bar{b}z+a=0$ maps the circle $|z|=1$ on the circle $|w|=1$, if $|a|\neq|b|$. Find the condition that the interior of first circle may be mapped on the interior of the second circle.

Answers

1. $f(0)=\infty, f(1)=i, f(i)=1, f(\infty)=0$
2. $f(0)=2i, f(1)=1+i, f(i)=\infty, f(\infty)=0$
3. $f(0)=-1, f(1)=i, f(i)=\infty, f(\infty)=1$
4. $f(0)=\infty, f(1)=0, f(i)=1+i, f(\infty)=1$
5. $|w|\geq 1$ and $u\geq\dfrac{1}{2}$
6. $v\geq 1$ and $|w|\geq 2$
7. $u\leq 0$ and $|w-1|\geq 2$
8. $|w-1|\geq 1$ and $v\geq\dfrac{1}{2}$
9. $v\geq 0$ and $\left|w-\dfrac{1}{2}\right|\geq\dfrac{1}{2}$
10. $u\geq 0$ and $v\geq 0$
11. $v\geq 0$ and $u\leq 1$
12. $u\leq 1$ and $\left|w-1-\dfrac{i}{2}\right|\geq\dfrac{1}{2}$
13. The image consists of a set of all points $w=u+iv$ such that $\left|w+\dfrac{1}{3}\right|\geq\dfrac{2}{3}$ and $v\leq 0$.
14. The image consists of a set of all points $w=u+iv$ such that $u+v\leq 0$ and $u+v+1\geq 0$.

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15. The image consists of a set of all points $w=u+iv$ such that $\left|w+\frac{1}{20}\right| \geq \frac{9}{20}$ and $u \geq -\frac{1}{2}$.

16. The image consists of a set of all points $w=u+iv$ such that $|w| \geq 1$ and $v \geq 0$.

18. $\left(-\frac{1}{2}, 0\right)$ 19. Centre $\left(\frac{1}{8}, \frac{1}{8}\right)$ and radius $\frac{1}{2\sqrt{2}}$

20. Centre $\left(\frac{1+e^{2a}}{2e^{a}}, 0\right)$ and radius $\frac{1-e^{2a}}{2e^{a}}$

5.3 Some properties of bilinear transformations

Def. Invariant or fixed points : The points which coincide with their transformations are called invariant or fixed points of the transformation i.e. fixed points of a transformation $w = f(z)$ are obtained by the equation $z = f(z)$. For the transformation $w = \frac{az+b}{cz+d}$, the fixed points are given by $w = z$ i.e. $z = \frac{az+b}{cz+d}$.

Result : In general, there are two fixed points but there is only one fixed point if $(a-d)^2 + 4bc = 0$. If there are two distinct fixed point *p* and *q*, then transformation may be put in the form $\frac{w-p}{w-q} = k\frac{z-p}{z-q}$ and if there is only one fixed point *p*, then transformation may be put in the form $\frac{1}{w-p} = \frac{1}{z-p} + K$.

Def. Critical points : Consider the bilinear transformation

$$w = f(z) = \frac{az+b}{cz+d} \qquad \text{.....(1)}$$

Solving this for *z*, we get the inverse map

$$z = f^{-1}(w) = \frac{b-wd}{wc-a} \qquad \text{.....(2)}$$

The transformation *f* associates a unique point of the *w*-plane to any point of *z*-plane except the point $z = -\frac{d}{c}$ $(c \neq 0)$. The transformation f^{-1} associates a unique point of the *z*-plane to any point of *w*-plane except the point $w = \frac{a}{c}$ $(c \neq 0)$. These exceptional points are mapped into the points $w = \infty$ and $z = \infty$ respectively which is clear from (1) and (2).

From (1), we have

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$$\frac{dw}{dz} = \frac{(cz+d).a-(az+b).c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

$$\Rightarrow \qquad \frac{dw}{dz} = \begin{cases} \infty & \text{if } z = -\dfrac{d}{c}, c \neq 0 \\ 0 & \text{if } z = \infty \end{cases}$$

The points $z = -\dfrac{d}{c}$ and $z = \infty$, where the conformal property does not hold are called critical points.

Def. Cross Ratio of the complex numbers z_1, z_2, z_3, z_4 is the complex number $\dfrac{(z_4 - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)}$.

Result : If $w = f(z)$ is a bilinear transformation that maps the distinct points z_1, z_2, z_3 and z_4 onto the distinct points w_1, w_2, w_3 and w_4 respectively, then

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)} = \frac{(w_4 - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w_4)}$$ *i.e* a bilinear transformation preserves the cross ratio.

Exercise 5.3

In problems 1–6, construct a linear fractional transformation that takes the given points z_1, z_2 and z_3 onto the given points w_1, w_2 and w_3 respectively :

1. $z_1 = -1$, $z_2 = 0$, $z_3 = 2$; $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$
2. $z_1 = i$, $z_2 = 0$, $z_3 = -i$; $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$
3. $z_1 = 0$, $z_2 = i$, $z_3 = \infty$; $w_1 = 0$, $w_2 = 1$, $w_3 = 2$
4. $z_1 = -1$, $z_2 = 0$, $z_3 = 1$; $w_1 = i$, $w_2 = 0$, $w_3 = \infty$
5. $z_1 = 1$, $z_2 = i$, $z_3 = -i$; $w_1 = -1$, $w_2 = 0$, $w_3 = 3$
6. $z_1 = 1$, $z_2 = i$, $z_3 = -i$; $w_1 = -i$, $w_2 = i$, $w_3 = \infty$

In problems 7 to 9, find the invariant (fixed) points of the given transformation :

7. $f(z) = \dfrac{5z+4}{z+5}$ 8. $f(z) = \dfrac{z-1}{z+1}$ 9. $f(z) = \dfrac{2iz+5}{z-2i}$

10. Let *a,b,c* and *d* be complex numbers such that $ad - bc \neq 0$.

    (a) Solve the equation $w = \dfrac{az+b}{cz+d}$ for z.

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(b) Explain why (a) implies that the linear fractional transformation $f(z)=\dfrac{(az+b)}{(cz+d)}$ is a one-to-one function.

11. Consider the equation $|z-a|=\lambda|z-b|$ where λ is a positive real constant.

(a) Show that the set of points satisfying the equation is a line if $\lambda=1$.

(b) Show that the set of points satisfying the equation is a circle if $\lambda\neq 1$

12. Let $T(z)=\dfrac{az+b}{cz+d}$ be a linear fractional transformation.

(a) If $T(0)=0$, then what, if anything, can be said about the coefficients $a,b,c,$ and d?

(b) If $T(1)=1,$ then what, if anything, can be said about the coefficients a,b,c and d?

(c) $T(\infty)=\infty,$ then what, if anything, can be said about the coefficients a,b,c and d?

13. Use problem 12 to show that if T is a linear fractional transformation and $T(0)=0, T(1)=1,$ and $T(\infty)=\infty$, then T must be the identity function. That is, $T(z)=z$.

14. If ∞ is a fixed point of a bilinear transformation then it must be of the form $f(z)=\alpha z+\beta$ where $\alpha\neq 0$.

15. If ∞ is the only fixed point of a bilinear transformation then it must be of the form $f(z)=z+\beta$ where $\beta\neq 0$.

16. If 0 and ∞ are the fixed points of a bilinear transformation then it must be of the form $f(z)=\alpha z$ where $\alpha\neq 0$.

17. If 0 and ∞ are the only fixed points of a bilinear transformation then it must be of the form $f(z)=\alpha z$ where $\alpha\neq 0,1$.

Answers

1. $T(z)=\dfrac{2z+2}{-z+2}$
2. $T(z)=\dfrac{i-z}{i+z}$
3. $T(z)=\dfrac{2z}{z+i}$
4. $T(z)=\dfrac{2iz}{z-1}$
5. $T(z)=\dfrac{3z-3i}{(1+4i)z-(4+i)}$
6. $T(z)=\dfrac{3(z-1)-i(z+1)}{(z+1)+i(1-z)}$
7. $z=\pm 2$
8. $z=\pm i$
9. $z=2i\pm 1$
10. (a) $z=\dfrac{b-dw}{cw-a}$

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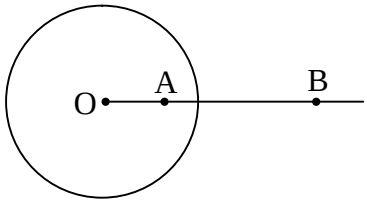
5.4 Evaluation of Bilinear Transformations

Def. Inverse points of a circle : Two points *A* and *B* are said to be inverse points of a circle with centre at '*O*' and radius '*r*' , if *A* and *B* are on the same side of centre *i.e* '*O*' and $(OA).(OB) = r^2$.



Results :

1. If a point is outside the circle then its inverse point will be inside the circle and vice-versa.
2. The inverse point of centre of circle is the point at infinity.
3. The inverse point of a point on the boundary is the point itself.
4. Inverse point of '*a*' w.r.t circle $|z - z_0| = r$ is given by $\text{Inv}(a) = z_0 + \dfrac{r^2}{\overline{a} - \overline{z_0}}$

Def. Symmetric points of a line : Two points *A* and *B* are said to be symmetric points of a line *L* if *L* is perpendicular bisector of the segment *AB*.

Three famous equations of a circle in complex plane :

(i) $z\bar{z} + b\bar{z} + \bar{b}z + c = 0$, where *c* is real constant and *b* may be complex.

(ii) $|z - z_0| = r$; here z_0 is the centre of the circle and the *r* is the radius.

(iii) $\left|\dfrac{z-p}{z-q}\right| = k$; $k \neq 1$ where *p* and *q* are inverse points of circle.

Two equations of line in complex plane :

(i) $b\bar{z} + \bar{b}z + c = 0$; where *c* is real constant and *b* may be complex.

(ii) $\left|\dfrac{z-p}{z-q}\right| = 1$; where *p* and *q* are symmetric points of the line.

Theorem : Four important theorems on Bilinear Transformation:-

1. If under a bilinear transformation line l_1 goes to line l_2 then symmetric points of l_1 goes to symmetric points of l_2.
2. If under a bilinear transformation line l goes to circle *C* then symmetric points of *l* goes to inverse points of *C*.

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3. If under a bilinear transformation circle C_1 goes to circle C_2 then inverse points of C_1 goes to inverse points of C_2.
4. If under a bilinear transformation circle C goes to line l then inverse points of C goes to symmetric points of l.

Results :

1. Symmetric point of 'z' w.r.t. x-axis is '$\bar{z}$'.
2. Symmetric point of 'z' w.r.t. y-axis is '$-\bar{z}$'.

Exercise 5.4

1. Find all bilinear transformations which maps $\text{Im}(z) \geq 0$ to $|w| \leq 1$ *i.e* which maps upper half plane to unit circle with centre at origin.
2. Find all bilinear transformations which maps $\text{Im}(z) \leq 0$ to $|w| \leq 1$ *i.e* which maps lower half plane to unit circle with centre at origin.
3. Find all bilinear transformations which transforms $\text{Re}(z) \geq 0$ to $|w| \leq 1$ *i.e* right half plane to unit circle.
4. Find all bilinear transformations which transforms $\text{Re}(z) \leq 0$ to $|w| \leq 1$ *i.e* left half plane to unit circle.
5. Find all bilinear transformations which maps $|z| \leq 1$ to $|w| \leq 1$ *i.e* unit circle to unit circle.
6. Find all bilinear transformations which maps $|z| \leq r_1$ to $|w| \leq r_2$.

Answers

1. $w = e^{i\theta}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right), \text{Im}(\alpha) > 0$

3. $w = e^{i\theta}\left(\frac{z-\alpha}{z+\bar{\alpha}}\right), \text{Re}(\alpha) > 0$

5. $w = e^{i\theta} z$ if $c = 0$ and $w = e^{i\theta}\left(\frac{z-\alpha}{\bar{\alpha} z - 1}\right), |\alpha| < 1$ if $c \neq 0$

6. $w = \frac{r_2}{r_1} e^{i\theta} z$ if $c = 0$ and $w = r_2 r_1 e^{i\theta}\left(\frac{z-\alpha}{\bar{\alpha} z - r_1^2}\right), |\alpha| < r_1$ if $c \neq 0$

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5.5 Conformal Mappings

Def. Conformal Mapping : Let $w=f(z)$ be a complex mapping defined in a domain D and let z_0 be a point in D. Then we say that $w=f(z)$ is conformal at z_0 if for every pair of smooth curves C_1 and C_2 in D intersecting at z_0 the angle between C_1 and C_2 at z_0 is equal to the angle between the image curves C_1' and C_2' at $f(z_0)$ in both magnitude and sense.

Def. Isogonal mapping : A mapping which preserves the magnitudes of angles but not necessarily the sense is called isogonal mapping.

Example : The mapping $w=\overline{z}$ is isogonal but not conformal.

Theorem 1 : Conformal mapping : If f is an analytic function in a domain D containing z_0, and if $f'(z_0)\neq 0,$ then $w=f(z)$ is a conformal mapping at z_0.

Def. Critical point : If a complex function f is analytic at a point z_0 and if $f'(z_0)=0,$ then z_0 is called a critical point of f.

Although it does not follow from Theorem 1, it is true that analytic functions are not conformal at critical points. More specifically, we can show that the following magnification of angles occurs at a critical point.

Theorem 2 : Angle Magnification at a Critical Point : Let f be analytic at the critical point z_0. If $n>1$ is an integer such that $f'(z_0)=f''(z_0)=\ldots=f^{(n-1)}(z_0)=0$ and $f^{(n)}(z_0)\neq 0,$ then the angle between any two smooth curves intersecting at z_0 is increased by a factor of n by the complex mapping $w=f(z).$ In particular, $w=f(z)$ is not a conformal mapping at z_0.

Exercise 5.5

In problems 1–14, determine where the complex mapping $w=f(z)$ is conformal.

1. $f(z)=z^3-3z+1$
2. $f(z)=z^2+2iz-3$
3. $f(z)=z-e^{-z}+1-i$
4. $f(z)=ze^{z^2-2}$
5. $f(z)=\tan z$
6. $f(z)=z-Ln(z+i)$
7. $f(z)=\dfrac{1}{z}$
8. $f(z)=\sin z$
9. $f(z)=\cos z$

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10. $f(z)=z+\frac{1}{z}$ 11. $f(z)=z+e^{-z}-5$ 12. $f(z)=ze^{z^3}+1$

13. $f(z)=z+az^2$ 14. $f(z)=z+az^3$

In problems 15-22 determine points where the complex mapping $w=f(z)$ fails to be conformal.

15. $f(z)=z^2$ 16. $f(z)=z^2-e^{\pi z^2}$ 17. $f(z)=z^5+1$ 18. $f(z)=\cosh z$

19. $f(z)=\sinh z$ 20. $f(z)=(z-i)^3$ 21. $f(z)=(iz-3)^2$ 22. $f(z)=e^{z^2}$

Answers

1. $\mathbb{C}-\{1,-1\}$ 2. $\mathbb{C}-\{-i\}$ 3. $\mathbb{C}-\{(2n+1)\pi i\}$ 4. $\mathbb{C}-\left\{\frac{1}{\sqrt{2}}i,-\frac{1}{\sqrt{2}}i\right\}$

5. $\mathbb{C}-\left\{(2n+1)\frac{\pi}{2}\right\}$ 6. $\mathbb{C}-\{\{z:x\le 0,\ y+1=0\}\cup\{1-i\}\}$ 7. $\mathbb{C}-\{0\}$

8. $\mathbb{C}-\left\{(2n+1)\frac{\pi}{2}\right\}$ 9. $\mathbb{C}-\{n\pi\}$ 10. $\mathbb{C}-\{0,1,-1\}$ 11. $\mathbb{C}-\{2n\pi i\}$

12. $\mathbb{C}-\left\{\left(-\frac{1}{3}\right)^{\frac{1}{3}},\left(-\frac{1}{3}\right)^{\frac{1}{3}}\omega,\left(-\frac{1}{3}\right)^{\frac{1}{3}}\omega^2\right\}$ 13. $\mathbb{C}$ if $a=0$ and $\mathbb{C}-\left\{-\frac{1}{2a}\right\}$ if $a\neq 0$

14. $\mathbb{C}$ if $a=0$ and $\mathbb{C}-\left\{\frac{i}{\sqrt{3a}},-\frac{i}{\sqrt{3a}}\right\}$ if $a\neq 0$ 15. 0 16. $0,\pm i\sqrt{\frac{\log\pi}{\pi}}$

17. 0 18. $n\pi i$ 19. $(2n+1)\frac{\pi i}{2}$ 20. i 21. $-3i$ 22. 0

5.6 Some more mappings

Results : The mappings $w=\sin z$ and $w=\cos z$

(i) maps the horizontal lines to ellipses.

(ii) maps the vertical lines to the hyperbolas.

Exercise 5.6

1. Find the image of $|z|=1$ and $x=0$ under the mapping $f(z)=\frac{1}{z}$.

2. Find the image of unit circle $|z|=1$ under the mapping $f(z)=z+\frac{1}{z}$.

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3. Find the image of $x=1$ and $y=1$ under the mapping $f(z)=z^2$.

4. Find the image of first quadrant i.e., $S=\{z:\mathrm{Re}(z)>0,\ \mathrm{Im}(z)>0\}$ under the mapping $w=z^2$.

5. Find the image of $x=1$, and $y=1$ under the mapping $f(z)=\sin z$.

6. Find the image of $x=1$ and $y=1$ under the mapping $f(z)=\cos z$.

Answers

1. $|w|=1$ and $u=0$ 2. $w=2\cos\theta, 0\le\theta\le 2\pi$ 3. $u=1-\frac{v^2}{4}$ and $u=\frac{v^2}{4}-1$

5. $\left(\frac{u}{\sin 1}\right)^2-\left(\frac{v}{\cos 1}\right)^2=1$ and $\left(\frac{u}{\cosh 1}\right)^2+\left(\frac{v}{\sinh 1}\right)^2=1$

6. $\left(\frac{u}{\cos 1}\right)^2-\left(\frac{v}{\sin 1}\right)^2=1$ and $\left(\frac{u}{\cosh 1}\right)^2+\left(\frac{v}{\sinh 1}\right)^2=1$

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True-false exercise

1. If $f(z)$ is analytic at a point z_0, then the mapping $w = f(z)$ is conformal at z_0.

2. The mapping $w = z^2 + iz + 1$ is not conformal at $z = -\frac{i}{2}$.

3. The mapping $w = z^2 + 1$ is not conformal at $z = \pm i$.

4. The mapping $w = \overline{z}$ fails to be conformal at every point in the complex plane.

5. A linear fractional transformation is conformal at every point in its domain.

6. The entire complex plane is mapped onto the real axis $v = 0$ by $w = z + \overline{z}$.

7. The entire complex plane is mapped onto the unit circle $|w| = 1$ by $w = \frac{z}{|z|}$.

8. A Mobius transformation which carries the upper half-plane onto the unit disk *D* such that $z = 2i$ is mapped onto $w = 0$ while $z = \infty$ is mapped onto $w = -1$ is $\frac{2i - z}{z + 2i}$.

9. The lines $x = 3$ and $x = -3$ are mapped onto to the same parabola by $w = z^2$.

10. The image of a circle under a linear fractional transformation is a circle.

11. The linear fractional transformation $T(z) = \frac{z - i}{z + i}$ maps the points $0, -1$ and i onto the points $-i, \infty$ and 0 respectively.

12. Given any three distinct points z_1, z_2 and z_3 there is a linear fractional transformation that maps z_1, z_2 and z_3 onto 0,1 and ∞.

13. The inverse of the linear fractional transformation $T(z) = (az + b)/(cz + d)$ is $T^{-1}(z) = (cz + d)/(az + b)$.

14. There exist transcendental entire functions having no fixed points.

15. The composition of two Mobius transformations is again a Mobius transformation.

16. The set *M* of all Mobius transformations $T_{abcd}(z) = \frac{az + b}{cz + d}$ with $ad - bc \neq 0,$ forms a group with respect to the composition as a binary operation.

17. A Mobius transformation $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ taking 0, 1, 6 into 2, 3, 4 respectively, is $f(z) = \frac{z + (2/3)}{z + (3/2)}$.

18. A Mobius transformation, which maps the upper half-plane $\{z : \operatorname{Im} z > 0\}$ onto itself and fixing 0, ∞ and no other points, must be of the form $T(z) = \alpha z$ for some $\alpha > 0$ and $\alpha \neq 1$.

19. A Mobius transformation, which maps the upper half-plane $\{z : \operatorname{Im} z > 0\}$ onto itself which fixes ∞ and no other points must be of the form $T(z) = z + \beta$ for some $\beta \neq 0$ with $\beta \in \mathbb{R}$.

20. Let *T* be a Mobius transformation such that *T* fixes ∞. Then *T* carries $\mathbb{R}$ onto itself iff $T(z) = \alpha z + \beta$ for some $\alpha \in \mathbb{R} \setminus \{0\}, \beta \in \mathbb{R}$.

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Assignment

---------- S C Q ----------

1. The inverse point of $1+i$ with respect to the circle $|z-1|=2$ is
   1. $1-i$
   2. $1+4i$
   3. $1+2i$
   4. $-1-i$

2. The inverse point of the point z with respect to the circle $|z|=r$ is
   1. $\frac{r^2}{z}$
   2. $\frac{r^2}{\bar{z}}$
   3. $\frac{r}{\bar{z}}$
   4. None of these

3. By the transformation $w=ze^{\frac{i\pi}{4}}$, the line $x=0$ is transformed into the line
   1. $v=-u$
   2. $v=u$
   3. $u+v=1$
   4. $v=0$

4. Under the transformation $w=z+1-i$, the image of the line $y=0$ in the z–plane is
   1. $v=1$
   2. $v=-1$
   3. $u=1$
   4. $u=0$

5. Critical points of $w=\frac{\alpha z+\beta}{\gamma z+\delta}, \alpha\delta-\beta\gamma\neq 0$ are
   1. $-\frac{\delta}{\gamma}$
   2. $-\frac{\delta}{\gamma}$ and ∞
   3. $-\frac{\delta}{\gamma}$ and 0
   4. None of these

6. The bilinear transformation which maps the points $z=1, z=0, z=-1$ of z-plane into $w=i, w=0, w=-i$ of w-plane respectively is
   1. $w=iz$
   2. $w=z$
   3. $w=i(z+1)$
   4. None of these

7. Under the transformation $w=(1+i)z+2-i$, the line $x=0$ is mapped into the line
   1. $u+v=1$
   2. $v-u=1$
   3. $u+2v=1$
   4. None of these

8. Under the transformation $w=\frac{1}{z}$, the image of the line $y=\frac{1}{4}$ in z–plane is
   1. circle $u^2+v^2+4v=0$
   2. circle $u^2+v^2=4$
   3. circle $u^2+v^2=2$
   4. None of the above

9. The mapping $w=z^2-2z-3$ is
   1. conformal within $|z|=1$
   2. not conformal at $z=1$
   3. not conformal at $z=-1$ and $z=3$
   4. conformal everywhere

10. The fixed points of the mapping $w=\frac{(5z+4)}{(z+5)}$ are
    1. 2, 2
    2. 2, –2
    3. –2, –2
    4. $\frac{-4}{5}, 5$

11. If $T_1(z)=\frac{z+2}{z+3}$ and $T_2(z)=\frac{z}{z+1}$, then $T_2T_1(z)$ is
    1. $z+2$
    2. $\frac{2}{2z+5}$
    3. $\frac{z+2}{2z+5}$
    4. None of these

12. The invariant point of the transformation $w=\frac{z-1}{z+1}$ are
    1. $z=i$
    2. $z=\pm i$
    3. $z=\frac{i}{2}$
    4. $z=\frac{-i}{2}$

13. If $T(z)=\frac{z+2}{z+3}$, then $T^{-1}(z)$ is
    1. $\frac{2-3z}{z+1}$
    2. $\frac{2-3z}{z-1}$
    3. $\frac{1}{z+3}$
    4. None of these

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14. The Mobius transformation maps the points $1-2i,\ 2+i,\ 2+3i,$ respectively into $2+2i, 1+3i, 4$ is

1. $\omega=\dfrac{i(1-z)}{1+z}$
2. $\omega=\dfrac{1-z}{1+z}$
3. $\omega=\dfrac{(20+18i)-(32+12i)z}{(29+17i)-(11-3i)z}$
4. $\omega=\dfrac{(20+18i)+(32-12i)z}{(29+17i)-(11+3i)z}$

15. For which of the following mappings the point ∞ is the only fixed point.

1. $f(z)=2z+3$
2. $f(z)=3z$
3. $f(z)=\dfrac{z-1}{z+1}$
4. $f(z)=z+2018i$

16. If $T_1(z)=\dfrac{z+2}{z+3}$ and $T_2(z)=\dfrac{z}{z+1}$, then $T_2^{-1}T_1(z)$ is equal to

1. $z+3$
2. $z+2$
3. $z+6$
4. $z-3$

17. If $T_1(z)=\dfrac{z+2}{z+3}$ and $T_2(z)=\dfrac{z}{z+1}$, then $T_1T_2(z)$ is equal to

1. $\dfrac{3z+2}{4z+2}$
2. $\dfrac{2}{2z+1}$
3. $\dfrac{3z}{4z-2}$
4. None of these

18. If $z_1\neq z_2\neq z_3\neq z_4$ in C_∞. Then, cross ratio (z_1, z_2, z_3, z_4) is a real number if z_1, z_2, z_3, z_4 lies on

1. triangle
2. a parabola
3. a circle
4. a hyperbola

19. The mapping $f(z)=\overline{z}$ is

1. conformal
2. isogonal
3. neither conformal nor isogonal
4. analytic

20. The transformation $w=\dfrac{az+b}{cz+d}$ is said to be normalized, if $ad-bc$ is equal to

1. 0
2. 1
3. 2
4. 3

21. Under the mapping $w=z+2-i,$ the image of line $y=0$ is

1. $\mathrm{Im}(w)=1$
2. $\mathrm{Im}(w)=-1$
3. $\mathrm{Re}(w)=1$
4. $\mathrm{Re}(w)=-1$

22. By the transformation $w=ze^{i\frac{\pi}{3}}$, the line $y=0$ in the z-plane is transformed into the line

1. $v=\sqrt{3}u$
2. $u=\sqrt{3}v$
3. $u+\sqrt{3}v=0$
4. $v+\sqrt{3}u=0$

23. The function $f(z)=z^2$ maps the first quadrant onto

1. itself
2. upper half plane
3. third quadrant
4. right half plane

24. An example of a map which is conformal in the whole of its domain of definition is

1. $f(z)=z^2$
2. $f(z)=e^{z+2}$
3. $f(z)=\cos z$
4. $f(z)=e^{z^2}$

25. The bilinear transformation $w=\dfrac{2z}{z-2}$ maps $\{z:|z-1|<1\}$ onto

1. $\{w:\mathrm{Re}(w)<0\}$
2. $\{w:\mathrm{Im}(w)>0\}$

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3. $\{w:\mathrm{Re}(w)>0\}$
4. $\{w:|w+2|<1\}$

26. If $f(z)=(z-2)^3 g(z)$, where $g(z)$ is analytic in $|z-2|<r$ and $g(2)\neq 0$, if angle at the vertex $z=2$ is $\frac{\pi}{3}$, then the angle at $f(2)$ is

1. $\frac{2\pi}{3}$ 2. π 3. $\frac{\pi}{2}$ 4. $\frac{\pi}{3}$

27. The cross ratio of the four points (z_1,z_2,z_3,z_4) is real if and only if the four points, lie on a
1. circle
2. straight line
3. circle and on a straight line
4. circle or on a straight line

28. The fixed points of $f(z)=\frac{2iz+5}{z-2i}$ are

1. $1\pm i$ 2. $1\pm 2i$
3. $2i\pm 1$ 4. $i\pm 1$

29. A Mobius transformation which transforms the upper half plane onto the lower half plane is

1. $w=\bar{z}$ 2. $w=\frac{z-i}{z+i}$
3. $w=\frac{1}{z}$ 4. $w=\frac{z+i}{z-i}$

30. Bilinear transformation, which maps the half plane $\mathrm{Im}(z)\geq 0$ onto the circular disc $|w|\leq 1$, is

1. $w=e^{i\lambda}\left(\frac{z-\alpha}{z-\alpha}\right)$ 2. $w=e^{i\lambda}\left(\frac{z-\alpha}{z+\alpha}\right)$
3. $w=e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$ 4. None of these

31. The image of $x=$ constant under the transformation $w=\sin z$ is

1. a parabola 2. a hyperbola
3. a circle 4. an ellipse

32. The mapping $f(z)=\sin z$ maps

1. The horizontal lines to circles.
2. The horizontal lines to ellipses.
3. The horizontal lines to hyperbolas.
4. The horizontal lines to vertical lines.

33. The Mobius transformation $w=\frac{az+b}{cz+d}$ maps the unit circles in the *w*-plane into straight line in the *z*-plane, if

1. $|a|=|b|$ 2. $|b|=|c|$
3. $|c|=|d|$ 4. $|d|=|a|$

34. The magnification factor of the conformal mapping $w=\sqrt{2}e^{\frac{\pi i}{4}}z+(1-2i)$ is

1. 1 2. 2 3. 3 4. $\sqrt{2}$

35. If a bilinear transformation $w=\frac{az+b}{cz+d}$ which has one fixed point at α and the other fixed point ∞, has the form $w-\alpha=\lambda(z-\alpha)$, then λ equal to

1. $\frac{d}{a}$ 2. $\frac{c}{b}$ 3. $\frac{a}{d}$ 4. $\frac{b}{c}$

36. Let $f(z)=(z-z_0)^{100}e^z$, then the mapping $w=f(z)$ magnifies the angle at the vertex z_0 by the factor *k*, where *k* is

1. 1 2. 100 3. 101 4. 0

37. A linear transformation which takes the triangle $\Delta(z)=(0,1,i)$ in the *z*-plane into the triangle $\Delta(w)=(-1,-i,i)$ in the *w*-plane is

1. $W=(1-i)z-1$ 2. $W=(1+i)z-1$
3. $W=(1-i)z+1$ 4. $W=(1+i)z+1$

38. Which of the following transformation is not conformal in a unit disc centered at origin

1. $e^{\sin z}$ 2. $e^{\log z}$
3. $e^{\sin z^2}$ 4. $e^{\tan z}$

39. The non constant transformation $w=\bar{z}$ is
1. both isogonal and conformal
2. isogonal but not conformal

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3. conformal but not isogonal
4. neither isogonal nor conformal

40. A transformation $w=\dfrac{5-4z}{4z-2}$ transforms the circle $|z|=1$ into a circle (in w-plane) whose centre is

1. $\left(\dfrac{1}{2},0\right)$ 2. $\left(-\dfrac{1}{2},\dfrac{1}{2}\right)$
3. $\left(-\dfrac{1}{2},0\right)$ 4. $\left(\dfrac{1}{2},-\dfrac{1}{2}\right)$

41. The bilinear transformation that maps the points $z_1=2,\ z_2=i,\ z_3=-2$ into the points $w_1=1,\ w_2=i,\ w_3=-1$ is

1. $w=\dfrac{3z-2i}{iz-6}$ 2. $w=\dfrac{3z-2i}{iz+6}$
3. $w=\dfrac{3z+2i}{iz-6}$ 4. $w=\dfrac{3z+2i}{iz+6}$

42. Critical points of transformation $w=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$ are

1. $-\infty,0$ 2. $0,\infty$
3. ±1 4. $\pm\infty$

43. The bilinear transformation w which maps the points $0,1,\infty$ in the z-plane onto the points $-i,\infty,1$ in the w-plane is

1. $\dfrac{z-1}{z+i}$ 2. $\dfrac{z-i}{z+1}$
3. $\dfrac{z+i}{z-1}$ 4. $\dfrac{z+1}{z-i}$

(GATE 2003)

44. The function $f(z)=z^2$ maps the first quadrant onto

1. itself 2. upper half plane
3. third quadrant 4. right half plane

(GATE 2002)

45. Let f be a bilinear transformation that maps -1 to 1, i to ∞ and $-i$ to 0. Then, $f(1)$ is equal to

1. -2 2. -1 3. iI 4. $-i$

(GATE 2008)

46. Let G_1 and G_2 be the images of the disc $\{z\in\mathbb{C}:|z+1|<1\}$ under the transformations $\omega=\dfrac{(1-i)z+2}{(1+i)z+2}$ and $\omega=\dfrac{(1+i)z+2}{(1-i)z+2}$, respectively. Then,

1. $G_1=\{w\in\mathbb{C}:\mathrm{Im}(w)<0\}$ and $G_2=\{w\in\mathbb{C}:\mathrm{Im}(w)>0\}$
2. $G_1=\{w\in\mathbb{C}:\mathrm{Im}(w)>0\}$ and $G_2=\{w\in\mathbb{C}:\mathrm{Im}(w)<0\}$
3. $G_1=\{w\in\mathbb{C}:|w|>2\}$ and $G_2=\{w\in\mathbb{C}:|w|<2\}$
4. $G_1=\{w\in\mathbb{C}:|w|<2\}$ and $G_2=\{w\in\mathbb{C}:|w|>2\}$

(GATE 2007)

47. Define $f:\mathbb{C}\to\mathbb{C}$ by

$$f(z)=\begin{cases}0, & \text{if } \mathrm{Re}(z)=0 \text{ or } \mathrm{Im}(z)=0,\\ z, & \text{otherwise}\end{cases}$$

Then, the set of points, where f is analytic is

1. $\{z:\mathrm{Re}(z)\neq0 \text{ and } \mathrm{Im}(z)\neq0\}$
2. $\{z:\mathrm{Re}(z)\neq0\}$
3. $\{z:\mathrm{Re}(z)\neq0 \text{ or } \mathrm{Im}(z)\neq0\}$
4. $\{z:\mathrm{Im}(z)\neq0\}$ **(GATE 2007)**

48. Let $w=f(z)$ be the bilinear transformation that maps -1, 0 and 1 to $-i$, 1 and i, respectively. Then, $f(1-i)$ equals

1. $-1+2i$ 2. $2i$
3. $-2+i$ 4. $-1+i$

(GATE 2004)

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49. Let $f=u+iv$ and $g=v+iu$ be non zero analytic functions on $|z|<1$. Then, it follows that
1. $f'\equiv 0$
2. f is conformal on $|z|<1$
3. $f\equiv kg$ for some real k
4. f is one to one

(GATE 2004)

50. Let $w(z)=\dfrac{az+b}{cz+d}$ and $f(z)=\dfrac{\alpha z+\beta}{\gamma z+\delta}$ be bilinear (Mobius) transformations. Then, the following is also a bilinear transformation
1. $f(z)w(z)$
2. $f(w(z))$
3. $f(z)+w(z)$
4. $f(z)+\dfrac{1}{w(z)}$

(GATE 2002)

51. The fixed points of $f(z)=\dfrac{2iz+5}{z-2i}$ are
1. $1\pm i$
2. $1\pm 2i$
3. $2i\pm 1$
4. $i\pm 1$

(GATE 2001)

52. The transformation $w=e^{i\theta}\left(\dfrac{z-p}{\bar{p}z-1}\right)$, where p is a constant, maps $|z|<1$ onto
1. $|w|<1$, if $|p|<1$
2. $|w|>1$, if $|p|>1$
3. $|w|=1$, if $|p|=1$
4. $|w|=3$, if $|p|=0$

(GATE 2000)

53. Let $a,b,c,d\in\mathbb{R}$ be such that $ad-bc>0$. Consider the Mobius transformation $T_{a,b,c,d}(z)=\dfrac{az+b}{cz+d}$. Define
$H_+=\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$,
$H_-=\{z\in\mathbb{C}:\mathrm{Im}(z)<0\}$
$R_+=\{z\in\mathbb{C}:\mathrm{Re}(z)>0\}$,
$R_-=\{z\in\mathbb{C}:\mathrm{Re}(z)<0\}$
Then, $T_{a,b,c,d}$ maps
1. H_+ to H_+
2. H_+ to H_-
3. R_+ to R_+
4. R_+ to R_-

(CSIR NET Dec 2015)

54. The set of all bilinear transformations under the product of transformations forms
1. a semi group
2. an abelian group
3. a non abelian group
4. None of these

55. Consider the map $\varphi:\mathbb{C}\setminus\{1\}\to\mathbb{C}$ given by $\varphi(z)=\dfrac{1+z}{1-z}$.
Which of the following is true ?
1. $\varphi\left(\{z\in\mathbb{C}\mid |z|<1\}\right)\subseteq\{z\in\mathbb{C}\mid |z|<1\}$
2. $\varphi\left(\{z\in\mathbb{C}\mid \mathrm{Re}(z)<0\}\right)\subseteq\{z\in\mathbb{C}\mid \mathrm{Re}(z)<0\}$
3. φ is onto
4. $\varphi(\mathbb{C}\setminus\{1\})=\mathbb{C}\setminus\{-1\}$

(CSIR NET June 2018)

---------- **MCQ** ----------

1. Define $H^+=\{z\in\mathbb{C}:y>0\}$
$H^-=\{z\in\mathbb{C}:y<0\}$
$L^+=\{z\in\mathbb{C}:x>0\}$
$L^-=\{z\in\mathbb{C}:x<0\}$
The function $f(z)=\dfrac{z}{3z+1}$
1. maps H^+ onto H^+ and H^- onto H^-
2. maps H^+ onto H^- and H^- onto H^+
3. maps H^+ onto L^+ and H^- onto L^-
4. maps H^+ onto L^- and H^- onto L^+

(CSIR NET June 2011)

2. Let U be an open subset of $\mathbb{C}$ containing $\{z\in\mathbb{C}:|z|\le 1\}$ and let $f:U\to\mathbb{C}$ be the map defined by $f(z)=e^{i\psi}\dfrac{z-a}{1-\bar{a}z}$ for $a\in D$ and $\psi\in[0,2\pi]$. Which of the following statements are true ?
1. $\left|f(e^{i\theta})\right|=1$ for $0\le\theta\le 2\pi$
2. f maps $\{z\in\mathbb{C}:|z|<1\}$ onto itself
3. f maps $\{z\in\mathbb{C}:|z|<1\}$ into itself
4. f is one one

(CSIR NET Dec 2011)

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3. For $z \in \mathbb{C}$ of the form $z = x + iy$, define
$H^+ = \{z \in \mathbb{C} : y > 0\}$,
$H^- = \{z \in \mathbb{C} : y < 0\}$,
$L^+ = \{z \in \mathbb{C} : x > 0\}$,
$L^- = \{z \in \mathbb{C} : x < 0\}$.
The function $f(z) = \dfrac{2z+1}{5z+3}$
1. maps H^+ onto H^+ and H^- onto H^-
2. maps H^+ onto H^- and H^- onto H^+
3. maps H^+ onto L^+ and H^- onto L^-
4. maps H^+ onto L^- and H^- onto L^+

(CSIR NET June 2012)

4. Let $f(z) = z + \dfrac{1}{z}$ for $z \in \mathbb{C}$ with $z \neq 0$.
Which of the following are always true ?
1. f is an analytic function on $\mathbb{C} \setminus \{0\}$.
2. f is a conformal map on $\mathbb{C} \setminus \{0\}$.
3. f maps the unit circle to a subset of the real axis.
4. The image of any circle in $\mathbb{C} \setminus \{0\}$ is again a circle.

(CSIR NET Dec 2012)

5. Let $f(z) = \dfrac{1+z}{1-z}$. Which of the following is/are true?
1. f maps $\{|z| < 1\}$ onto $\{\mathrm{Re}(z) > 0\}$.
2. f maps $\{|z| < 1, \mathrm{Im}(z) > 0\}$, onto $\{\mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$.
3. f maps $\{|z| < 1, \mathrm{Im}(z) < 0\}$ onto $\{\mathrm{Re}(z) < 0, \mathrm{Im}(z) < 0\}$.
4. f maps $\{|z| > 1\}$ onto $\{\mathrm{Im}(z) > 0\}$.

(CSIR NET Dec 2013)

6. Let $H = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the upper half plane and $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc. Suppose that f is a Mobius transformation, which maps H conformally onto D. Suppose that $f(2i) = 0$. Pick each correct statement from below.
1. f has a simple pole at $z = -2i$.
2. f satisfies $f(i)\overline{f(-i)} = 1$.
3. f has an essential singularity at $z = -2i$.
4. $|f(2+2i)| = \dfrac{1}{\sqrt{5}}$.

(CSIR NET June 2016)

7. Consider the Mobius transformation $f(z) = \dfrac{1}{z}$, $z \in \mathbb{C}$, $z \neq 0$. If C denotes a circle with positive radius passing through the origin, then f maps $C \setminus \{0\}$ to
1. a circle
2. a line
3. a line passing through the origin
4. a line not passing through the origin

(CSIR NET June 2017)

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True false key

1. F	2. T	3. F	4. T
5. T	6. T	7. F	8. T
9. F	10. F	11. T	12. T
13. F	14. T	15. T	16. T
17. F	18. T	19. T	20. T

Assignment key

S C Q

1. 2	2. 2	3. 1	4. 2
5. 2	6. 1	7. 1	8. 1
9. 2	10. 2	11. 3	12. 2
13. 2	14. 4	15. 4	16. 2
17. 4	18. 3	19. 2	20. 2
21. 2	22. 1	23. 2	24. 2
25. 1	26. 2	27. 4	28. 3
29. 3	30. 1	31. 2	32. 2
33. 3	34. 4	35. 3	36. 2
37. 1	38. 3	39. 2	40. 3
41. 4	42. 3	43. 3	44. 2
45. 2	46. 2	47. 1	48. 1
49. 1	50. 2	51. 3	52. 1
53. 1	54. 3	55. 4	

M C Q

1. 1	2. 1,2,3,4	3. 1
4. 1,3	5. 1,2	6. 1,2,4
7. 2,4		

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Chapter - 6

Some more topics

6.1 Maximum and minimum modulus principle

Maximum Modulus Principle : Let D be a domain and ∂D denotes its boundary. If a function f is non-constant and analytic on $D \cup \partial D$ then $|f(z)|$ attains its maximum on ∂D.

Minimum Modulus Principle : Let D be a domain and ∂D denotes its boundary. If a function f is non constant and analytic on $D \cup \partial D$ and $f(z) \neq 0$ for all $z \in D$ then $|f(z)|$ attains its minimum on ∂D.

Maximum modulus theorem : Let D be a domain and ∂D denotes its boundary. If a function f is non-constant and analytic on D and continuous on $D \cup \partial D$ then $|f(z)|$ attains its maximum on ∂D.

Minimum modulus theorem : Let D be a domain and ∂D denotes its boundary. If a function f is non-constant and analytic on D, $f(z) \neq 0$ for all $z \in D$ and continuous on $D \cup \partial D$ then $|f(z)|$ attains its minimum on ∂D.

Remarks :

1. The function $f(z) = z^2$ is analytic and non-constant on $|z| \leq 1$ and minimum value of $|f(z)|$ is attained on the point $z = 0$, which is not a point of the boundary. Is this a contradiction to minimum modulus principle.
2. **Another version of maximum modulus principle :** If a function f is analytic within and on the boundary of a domain D and $|f(z)|$ assumes the maximum value at some interior point of D, then f must be constant.

Some Results :

1. Maximum modulus of $f(z) = az + b$ on the circular region $|z| \leq 1$ is $|a| + |b|$ and this maximum is attained at the point $z = e^{i\theta}$ where $\theta = \arg(b) - \arg(a)$.
2. Maximum modulus of $f(z) = az + b$ on the circular region $|z| \leq r$ is $|a|r + b$ and this maximum is attained at the point $z = re^{i\theta}$ where $\theta = \arg(b) - \arg(a)$.
3. Maximum modulus of $f(z) = az + b$ on the circular region $|z - z_0| \leq r$ is $|a|(|z_0| + r) + |b|$.

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4. Maximum modulus of $f(z)=az^n+b$ on the circular region $|z|\le 1$ is $|a|+|b|$.

5. Maximum modulus of $f(z)=e^z$ on the circular region $|z-z_0|\le r$ is $e^{Re(z_0)+r}$ and this maximum is attained at the point z_0+r.

Exercise 6.1

Find the maximum modulus and its point of attainment of the following function on the indicated region :

(1) $f(z)=2z+5i$ on $|z|\le 1$

(2) $f(z)=-iz+i$ on $|z|\le 5$

(3) $f(z)=z^2+4z$ on $|z|\le 1$

(4) $f(z)=z^3+3z-1$ on $|z|\le 1$

(5) $f(z)=z^2$ on $|z-1-i|\le 1$

(6) $f(z)=(iz+3)^2$ on $|z|\le 1$

(7) $f(z)=-2iz^2+5$ on $|z|\le 1$

(8) $f(z)=\left(z-2-2\sqrt{3}i\right)^2$ on $|z|\le 1$

Answers

1. 7 at $z=i$
2. 6 at $z=-5$
3. 5 at $z=1$
4. 5 at $z=-1$
5. $3+2\sqrt{2}$
6. 16 at $z=-i$
7. 7
8. 25

6.2 Schwartz Lemma

Schwarz Lemma : Let $D=\{z\in C:|z|<1\}$ and $f:D\to \bar{D}$ be an analytic function such that $z=0$ is a simple zero of $f(z)$ i.e., $f(0)=0,\ f'(0)\ne 0$.

Then

(i) $|f(z)|\le|z|$ for all $z\in D$.

(ii) $|f'(0)|\le 1$.

Further, inequality holds either in (i) for some $z_0\ne 0$ or in (ii) iff $f(z)=cz$ with $|c|=1$ i.e., f is a rotation.

Proof : Let $f(z)=a_0+a_1z+a_2z^2+.....$ be the Taylor's series expansion of $f(z)$ in D. Now,

$f(0)=0$ gives $a_0=0$ so that $f(z)=a_1z+a_2z^2+a_3z^3+.....$

Here, $f'(0)=a_1$. Now, we define a function g on D by $g(z)=\begin{cases}\dfrac{f(z)}{z} & \text{for} \quad z\ne 0,\ z\in D\\ f'(0)=a_1 & \text{for} \quad z=0\end{cases}$

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Then, clearly, g is analytic on $D-\{0\}$ and continuous on D, so by Riemann's removability theorem, g must be analytic on D. Now, we claim that $|g(z)|\le 1$ for all $z\in D$. Let $0<r<1$ and $D_r=\{z:|z|<r\}$, then g is analytic on D_r and continuous on its closure $\bar{D}_r=\{z:|z|\le r\}$. Therefore, maximum modulus theorem is applicable. Now $|f(z)|\le 1$ for all $z\in D$ so for $|z|=r$, we have $|g(z)|=\dfrac{|f(z)|}{|z|}\le\dfrac{1}{r}$ i.e., $\dfrac{1}{r}$ is an upper bound of $|g(z)|$ on $|z|=r$. So by maximum modulus theorem, we obtain $|g(z)|\le\dfrac{1}{r}$ for all $|z|\le r$. Since r is arbitrary, by letting $r\to 1$, we obtain that $|g(z)|\le 1$ for all $z\in D \Rightarrow \left|\dfrac{f(z)}{z}\right|\le 1$ for all $z\in D \Rightarrow |f(z)|\le|z|$ for all $z\in D$ which proves (i).

To prove (ii), we see that $|g(z)|\le 1$ for all $z\in D$. Taking $z=0$, we have $|g(0)|\le 1 \Rightarrow |f'(0)|\le 1$. Further, equality holds in (i) for some $z_0\neq 0,\ z_0\in D$ implies that $|f(z_0)|=|z_0|$ i.e., $|g(z_0)|=1$. It follows that g attains its maximum modulus at an interior point z_0. Therefore, by maximum modulus theorem, g must be a constant function, say $g(z)=c$ for all $z\in D$. Then, $|g(z_0)|=1$ gives $|c|=1$ and so $f(z)=cz$ with $|c|=1$.

Also, equality holds in (ii) implies that $|f'(0)|=1$ i.e., $|g(0)|=1$, which shows that g attains its maximum at interior point $z=0$. By above reasoning, we obtain the required result at once.

General version of Schwartz Lemma : Let $f(z)$ be analytic in $|z-a|<r$ such that $f(z)$ has a zero of order n at $z=a$ and $|f(z)|\le M$ for all z in $|z-a|<r$. Then

(i) $|f(z)|\le\dfrac{M|z-a|^n}{r^n}$ for all z in $|z-a|<r$.

(ii) $\left|f^{(n)}(a)\right|\le\dfrac{M\lfloor n}{r^n}$

Further equality holds in (i) (for some $z\neq a$) or in (ii) if and only if $f(z)$ is of the form $f(z)=\dfrac{M.c(z-a)^n}{r^n}$ where $|c|=1$.

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Schwartz Pick Lemma : Let $f(z)$ be analytic in $D=\{z:|z|<1\}$ such that $|f(z)|\le 1$ for all z in D.

Then (i) $|f'(z)|\le \dfrac{1-|f(z)|^2}{1-|z|^2}$ for all z in D.

(ii) $\left|\dfrac{f(z)-f(a)}{1-\overline{f(a)}\,f(z)}\right|\le\left|\dfrac{z-a}{1-\bar{a}z}\right|$ for all a and z in D.

(iii) $|f(z)|\le \dfrac{|f(0)|+|z|}{1+|f(0)||z|}$ for all z in D.

Remark : Schwartz Lemma is a special case of Schwartz Pick Lemma.

Exercise 6.2

1. Prove that there exists no function f that is analytic in D *s.t.* $f\left(\frac{1}{4}\right)=0$ and $f\left(-\frac{3}{4}\right)=\frac{17}{20}$.

2. Prove that there exists no function f that is analytic in D such that $|f(z)|\le 1$, $f\left(\frac{1}{3}\right)=0$ and $f\left(-\frac{2}{3}\right)=\frac{5}{6}$.

3. Find the estimate of $f\left(\frac{1}{4}\right)$ where f is an analytic function from D to D *s.t.* $f\left(\frac{1}{2}\right)=0$.

4. Prove that there exists no function $f:D\to\overline{D}$ *s.t.* $f\left(\frac{1}{2}\right)=0$ and $f\left(-\frac{1}{2}\right)=\frac{19}{20}$.

5. Prove that the existence of the function is guaranteed in above question if $\frac{19}{20}$ is replaced by $\frac{4}{5}$.

6. Show that there exists an analytic function $f:D\to\overline{D}$ *s.t.* $f\left(\frac{1}{3}\right)=0$ and $f\left(-\frac{1}{3}\right)=\frac{3}{5}$.

7. Prove that there exists an analytic function $f:D\to D$ *s.t.* $f\left(\frac{1}{2}\right)=0$ and $f'\left(\frac{1}{2}\right)\le\frac{4}{3}$.

8. Prove that there exists an analytic function $f:D\to D$ *s.t.* $f(0)=\frac{1}{2}$ and $f'(0)=\frac{3}{4}$.

9. Prove that there exists no function $f:D\to\overline{D}$ *s.t.* $f\left(\frac{1}{2}\right)=\frac{3}{4}$ and $f'\left(\frac{1}{2}\right)=\frac{3}{4}$.

10. Does there exists an analytic function $f:D\to\overline{D}$ with $f(0)=-\frac{1}{2}$ and $f'(0)=\frac{3}{4}$.

11. Does there exists an analytic function $f:D\to\overline{D}$ with $f(0)=\pm\frac{1}{2}$ and $f'(0)=\frac{4}{5}$.

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6.3 Rouche's theorem

Rouche's theorem : Let $f(z)$ and $g(z)$ are analytic within and on a simple closed contour *C* such that $|g(z)|<|f(z)|$ for all *z* on *C* (*i.e.* boundary) then $f(z)$ and $f(z)+g(z)$ has same number of zeros inside *C*, counting multiplicity.

Exercise 6.3

1. Determine the number of zeros, counting multiplicities, of the polynomial
   (i) $z^6-5z^4+z^3-2z$ (ii) $2z^4-2z^3+2z^2-2z+9$ (iii) $z^7-4z^3+z-1=0$
   inside the circle $|z|=1$.
2. Determine the number of zeros, counting multiplicities of the polynomial
   (i) z^4+3z^3+6 (ii) $z^4-2z^3+9z^2+z-1$ (iii) $z^5+3z^3+z^2+1$
   inside the circle $|z|=2$.
3. Determine the number of roots, counting multiplicities, of the equation $2z^5-6z^2+z+1=0$ in the annulus $1\le|z|<2$.
4. Show that the polynomial z^6-5z^4+7 has no zeros in $|z|<1$, four zeros in $|z|<2$ and all the six zeros in $|z|<3$.
5. Show that all the seven zeros of the polynomial z^7+10z^3+14 lie within the annulus $1<|z|<2$.
6. Show that the polynomial $4z^4+2(1-i)z+1$ has one zero in $|z|<\frac{1}{2}$ and three zeros in the annulus $\frac{1}{2}<|z|<1$.
7. Show that all the roots of the equation $z^4+6z+3=0$ are inside the circle $|z|=2$.
8. Show that if *c* is a complex number *s.t.* $|c|>e$, then the equation $cz^n=e^z$ has *n* roots, counting multiplicities, inside the circle $|z|=1$.
9. How many roots of the equation $6z^4+z^3-7z^2-z+1=0$ lie in the disc $|z|<1$. **[NET Dec.2010]**

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Answers

1. (i) 4 (ii) 0 (iii) 3 2. (i) 3 (ii) 2 (iii) 5 3. 3 9. 2

6.4 Argument Principle

Argument principle : Let C be a simple closed contour lying entirely within a domain D. Suppose f is analytic in D except at a finite number of poles inside C, and that $f(z) \neq 0$ on C. Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P$$

where N is the total number of zeros of f inside C and P is the total number of poles of f inside C (counting multiplicities).

Exercise 6.4

In problems 1-2, evaluate the integral $\oint_C \frac{f'(z)}{f(z)}dz$ for the given function f and closed contour C :

1. $f(z) = z^6 - 2iz^4 + (5-i)z^2 + 10$, C encloses all zeros of f

2. $f(z) = \frac{(z-3iz-2)^2}{z(z^2-2z+2)^5}$, C is $|z| = \frac{3}{2}$

In problems 3-6 , evaluate the given integral on the indicated closed contour C:

3. $\oint_C \frac{2z+1}{z^2+z}dz$, C is $|z| = 2$ 4. $\oint_C \frac{z}{z^2+4}dz$, C is $|z| = 3$

5. $\oint_C \cot z\, dz$, C is the rectangular contour with vertices $10+i, -4+i, -4-i$ and $10-i$

6. $\oint_C \tan \pi z\, dz$, C is $|z-1| = 2$

Answers

1. $12\pi i$ 2. $-18\pi i$ 3. $4\pi i$ 4. $2\pi i$ 5. $10\pi i$ 6. $-8i$

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6.5 Constant function theorem

Little Picard theorem : A non-constant entire function can omit atmost one value.

OR

If an entire function omit two or more values, then it becomes constant function.

Result : (i) $e^{\mathbb{C}} = \mathbb{C} - \{0\}$, i.e., the function $f(z) = e^z$ omit the number zero.

(ii) $\sin \mathbb{C} = \mathbb{C}, \cos \mathbb{C} = \mathbb{C}, \sinh \mathbb{C} = \mathbb{C}, \cosh \mathbb{C} = \mathbb{C}$, i.e., $\sin z, \cos z, \sinh z$ and $\cosh z$ omit no complex number.

Constant function theorem : Let $f(z) = u(x,y) + iv(x,y)$ be an entire function. Then if any one of the following conditions is satisfied then $f(z)$ will be a constant function.

1. $f'(z) = 0$
2. $\mathrm{Re}(f(z))$ is constant
3. $\mathrm{Im}(f(z))$ is constant
4. $|f(z)|$ is constant
5. $\mathrm{Arg}(f(z))$ is constant
6. $\mathrm{Re}(f(z))$ is bounded
7. $\mathrm{Im}(f(z))$ is bounded
8. $\mathrm{Re}(f(z)) \geq k$
9. $\mathrm{Im}(f(z)) \geq k$
10. $\mathrm{Re}(f(z)) \leq k$
11. $\mathrm{Im}(f(z)) \leq k$
12. $|f(z)|$ is harmonic
13. $\overline{f(z)}$ is analytic in D
14. $u = v^2$
15. $v = u^2$
16. $au + bv = c$; where a,b and c are complex constant
17. If $u(x,y)$ and $v(x,y)$ are harmonic conjugate of each other.

Open mapping theorem : A non-constant analytic function maps open set to a open set.

or

If image of an open set is not open under an analytic function then the function is a constant function.

or

Let f be a non-constant analytic function on a domain D. Let x_0 be an interior point of D then $f(x_0)$ is an interior point of $f(D)$.

Corollary : If f is an analytic function in a domain D and $f(D)$ lies on a curve then function is a constant function.

Result : A non-constant entire function can have atmost countable zeros. In other words, a non-constant entire function has either finitely many zeros or denumerable zeros.

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6.6 Some more theorems

Extended Liouville's theorem : Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f(z)| \le A + B|z|^k$; $A, B > 0$, $k \in \mathbb{N} \cup \{0\}$ $\left(|f(z)| \le A|z|^k\right)$ for sufficient large $\mathbb{Z}$, then f is a polynomial of degree atmost *k*.

Riemann's Removability theorem : Let f be a continuous function a domain $D \subseteq \mathbb{C}$ and analytic on D except the points of line segment (or a fine curve in D) then f is analytic in D.

Picard's theorem for meromorphic functions : A non constant meromorphic function can omit atmost two points.

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True-false exercise

1. There exist an analytic function f defined on $\mathbb{C}\setminus\{0\}$ such that $f'(z)=1/z$

2. There exists an analytic function f of Δ onto itself such that $f(0)=1/2,\ f(1/2)=1/3$ and $f(1/3)=1/4$.

3. There exists an analytic function $f:\Delta\to\Delta$ such that $f(1/2)=0$ and $|f'(1/2)|\le 4/3$

4. There exists an analytic function $f:\Delta\to\Delta$ such that $f(0)=1/2$ and $f'(0)=3/4$.

5. There exists an analytic function $f:\Delta\to\Delta$ such that $f(1/2)=3/4$ and $f'(1/2)=3/4$.

6. If f and g are two entire functions such that $e^f+e^g=1$, then f and g are constant functions.

7. If f is entire and $\operatorname{Re} f(z)$ is bounded as $|z|\to\infty$, then f is constant.

8. Suppose f is entire which takes real z into real and purely imaginary into purely imaginary. Then f is an odd function.

9. If f is entire such that f omits a non-empty disk, then f is constant.

10. An entire function f whose imaginary part is the square of the real part is constant in $\mathbb{C}$.

11. If $k\neq 1$ is a fixed constant and f is entire such that $f(z)=f(kz)$ for all $z\in\mathbb{C}$, then f is constant on $\mathbb{C}$.

12. If $f=u+iv$ is entire and $u^2\le v^2+2004$ on $\mathbb{C}$, then f is a constant.

13. All the roots of the polynomial $p(z)=1+z+z^2+z^3+z^4$ have absolute value less than 2.

14. If $f=u+iv$ is analytic in a domain D such that $au+bv$ is constant, where a, b are real constants (not both zero), then f is constant.

15. If f is analytic in $\mathbb{C}_\infty$ except for a finite number of poles, then the number poles and zeros of f (counted according to multiplicity) are equal.

16. $\displaystyle\int_{|z|=1}\frac{1}{e^z-1-2z}dz=-2\pi i$.

Assignment

---------- S C Q ----------

1. The number of zeros, counting multiplicities, of the polynomial $z^5+3z^3+z^2+1$ inside the circle $|z|=2$ is

   1. 0 2. 2 3. 3 4. 5

   (GATE 2004)

2. If $f(z)=z^5-3iz^2+2z+i-1$ and C encloses zeros of $f(z)$, then $\int_C\frac{f'(z)}{f(z)}dz$ is

   1. $5\pi i$
   2. 0
   3. $10\pi i$
   4. None of these

3. Let f be a polynomial function on the entire complex plane, such that $f(z)\neq 0$ for z such that $|z|<1$. Then, $\frac{1}{2\pi i}\int_{|z|=1}\frac{f'(z)}{f(z)}dz$

   1. can take only integer values
   2. can take any value
   3. is zero
   4. is equal to degree of f

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4. The value of $\int_C \frac{f'(z)}{f(z)}dz$, where

$f(z)=\frac{(z^2+1)^2}{(z^2+3z+2)^3}$ and C is the circle $|z|=3$ with positive sense, is

1. $2\pi i$ 2. $-2\pi i$
3. $4\pi i$ 4. $-4\pi i$

5. The number of roots of the equation $z^5-12z^2+14=0$ that lie in the region $\left\{z\in C: 2\le|z|<\frac{5}{2}\right\}$ is

1. 2 2. 3 3. 4 4. 5

6. Let T be the closed unit disk and ∂T be the unit circle. Then, which one of the following holds for every analytic function $f: T\to\mathbb{C}$.

1. $|f|$ attains its minimum and its maximum of ∂T
2. $|f|$ attains its minimum on ∂T but need not attain its maximum on ∂T
3. $|f|$ attains its maximum on ∂T but need not attain its minimum on ∂T
4. $|f|$ need not attain its maximum on ∂T and also it need not attain its minimum on ∂T

(GATE 2008)

7. Let $f(z)=2z^2-1$. Then, the maximum value of $|f(z)|$ on the unit disc $D=\{z\in C:|z|\le 1\}$ equals to

1. 1 2. 2 3. 3 4. 4

(GATE 2007)

8. Let $f(z)$ be defined on the domain $E:|z-2i|<3$ and on its boundary ∂E. Then, which one of the following statements is always true ?

1. If $f(z)$ is analytic on E and $f(z)\neq 0$ for any z in E, then $|f|$ attains its maximum on ∂E
2. If $f(z)$ is analytic on $E\cup\partial E$, then $|f|$ attains its minimum on ∂E
3. If $f(z)$ is analytic on E and continuous on $E\cup\partial E$, then $|f|$ attains its maximum and minimum on ∂E
4. If $f(z)$ is analytic on $E\cup\partial E$ and $f(z)\neq 0$ for any z in $E\cup\partial E$, then $|f|$ attains its minimum on ∂E

(GATE 2004)

9. Let $f: D\to D$ be a holomorphic function with $f(0)=0$, where D is the open unit disc $\{z\in\mathbb{C}:|z|<1\}$. Then

1. $|f'(0)|=1$ 2. $\left|f\left(\frac{1}{2}\right)\right|\le\frac{1}{2}$
3. $\left|f\left(\frac{1}{2}\right)\right|\le\frac{1}{4}$ 4. $|f'(0)|\le\frac{1}{2}$

(CSIR NET Dec 2012)

10. Let a,b,c be non collinear points in the complex plane and let Δ denote the closed triangular region of the plane with vertices a,b,c. For $z\in\Delta$, let $h(z)=|z-a|\cdot|z-b|\cdot|z-c|$. The maximum value of the function h :

1. is not attained at any point of Δ.
2. is attained at an interior point of Δ.
3. is attained at the centre of gravity of Δ.
4. is attained at a boundary point of Δ.

(CSIR NET Dec 2013)

11. Let $p(z), q(z)$ be two non zero complex polynomials. Then $p(z)\overline{q(z)}$ is analytic if and only if

1. $p(z)$ is constant
2. $p(z)q(z)$ is constant
3. $q(z)$ is a constant
4. $\overline{p(z)}q(z)$ is a constant

(CSIR NET June 2013)

12. Let f be a non constant entire function. Which of the following properties is possible for f for each $z\in\mathbb{C}$?

1. $\text{Re } f(z)=\text{Im } f(z)$.
2. $|f(z)|<1$.
3. $\text{Im } f(z)<0$.
4. $f(z)\neq 0$.

(CSIR NET Dec 2013)

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13. If G is a region and f is non constant analytic function on G. Then, open mapping theorem states, for any open set $U \subset G$
    1. $f(U)$ is closed
    2. $f(U)$ is open
    3. $f(U)=U$
    4. None of these

14. If $f(z)=u+iv$ is analytic and (u,v) lies on a circle of unit radius with centre origin for all values of $z \in \mathbb{C}$, then
    1. $f(z)$ has countably infinite singularity
    2. $f(z)$ is non constant entire function
    3. $f(z)$ is meromorphic function with non empty set of singularity
    4. $f(z)$ is constant

15. It D is the open unit disk in $\mathbb{C}$ and $f:\mathbb{C}\to D$ is analytic with $f(10)=\frac{1}{2}$, then $f(10+i)$ is
    1. $\frac{1+i}{2}$
    2. $\frac{1-i}{2}$
    3. $\frac{1}{2}$
    4. $\frac{i}{2}$

16. Let $f:C\to C$ s.t. $f(z)=e^z$, then
    1. $f(z)$ is univalent but not onto
    2. $f(z)$ is univalent in $|z|<a$ for $a>\pi$ but not onto
    3. $f(z)$ is univalent in $|z|<\pi$ and onto on C
    4. $f(z)$ is univalent in $|z|<\pi$ but not onto therein

17. Let $f=u+iv$ be analytic in a connected open set D where u and v are real valued. If u and v satisfies the condition $2u+3v=1$, then
    1. $f'(0)=1$
    2. $f'(0)=0$
    3. $f'(0)=2$
    4. $f'(0)=3$

18. Let f be an entire function and satisfies the following two equations $f(z+1)=f(z)$, $f(z+i)=f(z)$ for every z in C, then
    1. $f'(z)=f(z)$
    2. $f(z)\in C$
    3. $f \equiv$ constant
    4. f is non constant polynomial

19. The origin and the points representing the roots of the equation $z^2+pz+q=0$ form an equilateral triangle if
    1. $p=3q^2$
    2. $p=3q$
    3. $p^2=q$
    4. $p^2=3q$

20. The number of roots of the equation $z^8-4z^5+z^2-1=0$ that lie inside the circle $|z|=1$ (Using the Rouche's theorem) is
    1. 8
    2. 5
    3. 6
    4. None of these

21. If $f(z)=z^2+2$, then the minimum value of $|f(z)|$ over the closed region $|z|\le 1$ is
    1. 0
    2. 1
    3. 3
    4. 2

22. Let $h(z)$ is analytic in the whole plane, $h(0)=3+4i$ and $|h(z)|\le 5$ in $|z|<1$. Then, choose the incorrect statement
    1. $h(z)$ is bounded
    2. $h(0)=0$
    3. $h(z)$ is analytic at $z=\infty$
    4. $h(z)$ is non constant polynomial

23. Let $f(z)$ be an entire function such that for some constant $\alpha, |f(z)|\le \alpha|z|^3$ and $|z|\ge 1$ and $f(z)=f(iz)$ for all $z\in\mathbb{C}$, then
    1. $f(z)=\alpha z^3$, $\forall z\in\mathbb{C}$
    2. $f(z)$ is a constant
    3. $f(z)$ is a quadratic polynomial
    4. No such $f(z)$ exists

24. If f is an entire function, then
    1. f has a power series expansion
    2. f has not a power series expansion
    3. f is constant
    4. None of above

25. Let $f(z)$ be analytic on $D=\{z\in:|z-1|<1\}$ such that $f(1)=1$. If $f(z)=f(z^2), \forall z\in D$,

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then which one of the following statements is not correct ?

1. $f(z)=\left[f(z)\right]^2, \forall z \in D$
2. $f\left(\frac{z}{2}\right)=\frac{1}{2}f(z), \forall z \in D$
3. $f\left(z^3\right)=\left[f(z)\right]^3, \forall z \in D$
4. $f'(1)=0$

(GATE 2010)

26. Let S be the open unit disk and $f: S \to \mathbb{C}$ be a real valued analytic function with $f(0)=1$. Then, the set $\{z \in S : f(z) \neq 1\}$ is

1. empty
2. non empty finite
3. coutably infinite
4. uncountable

(GATE 2008)

27. Let $S=\{0\} \cup \left\{\frac{1}{4n+7} : n=1,2,...\right\}$. Then, the number of analytic functions which vanish only on S is

1. infinite
2. 0
3. 1
4. 2

(GATE 2007)

28. Let $f(z)$ be an entire function such that $|f(z)| \le K|z|, \forall z \in \mathbb{C},$ for some $K>0$. If $f(1)=i,$ the value of $f(i)$ is

1. 1
2. –1
3. i
4. $-i$

(GATE 2011)

29. Consider the functions $f, g: \mathbb{C} \to \mathbb{C}$ defined by $f(z)=e^z, g(z)=e^{iz}$. Let $S=\{z \in \mathbb{C} : \operatorname{Re} z \in [-\pi, \pi]\}$. Then

1. f is an onto entire function.
2. g is a bounded function on $\mathbb{C}$.
3. f is bounded on S.
4. g is bounded on S.

(CSIR NET Dec 2012)

30. Let f be a non constant holomorphic function in the unit disc $\{|z|<1\}$ such that $f(0)=1$. Then it is necessary that

1. there are infinitely many points z in the unit disc such that $|f(z)|=1$.
2. f is bounded.
3. there are at most finitely many points z in the unit disc such that $|f(z)|=1$.
4. f is a rational function.

(CSIR NET Dec 2013)

31. Suppose f and g are entire functions and $g(z) \neq 0$ for all $z \in \mathbb{C}$. If $|f(z)| \le |g(z)|$, then we conclude that

1. $f(z) \neq 0$ for all $z \in \mathbb{C}$.
2. f is a constant function.
3. $f(0)=0$.
4. for some $C \in \mathbb{C}$, $f(z)=Cg(z)$.

(CSIR NET Dec 2016)

32. Let f and g be two entire functions such that $\operatorname{Re}(f(z)) \le k \operatorname{Re} g(z) \quad \forall \ z \in \mathbb{C}$ then which of following is true

1. f and g are constant functions
2. $f(z)=a\,g(z)+b$
3. $g(z)=e^{f(z)}$
4. None of these

33. Let f is an entire function and $|f(z)| \le A+B|z|^{\frac{3}{2}}$ then

1. f is constant
2. f is a linear polynomial
3. f is quadratic polynomial
4. f can't be entire

34. Let f be an entire function such that $|f'(z)| \le |z|$ then

1. $f(z)=a+bz$
2. $f(z)=a+bz^2$; $|b| \le \frac{1}{2}$
3. f is constant
4. $f(z)$ is a polynomial of degree ≥ 3

35. Suppose that f is a non-constant analytic function defined over $\mathbb{C}$. Then which one of the following is false ?

1. f is unbounded
2. f sends open sets into open sets
3. There exists an open connected domain

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U on which f is never zero but $|f/U|$ attains its minimum at some point of U

4. The image of f is dense in $\mathbb{C}$

(CSIR NET June 2018)

----------**MCQ**----------

1. Let $f: D \to D$ be holomorphic with $f(0)=\frac{1}{2}$ and $f\left(\frac{1}{2}\right)=0$, where $D=\{z:|z|\le 1\}$. Which of the following is correct ?
   1. $|f'(0)| \le \frac{3}{4}$
   2. $\left|f'\left(\frac{1}{2}\right)\right| \le \frac{4}{3}$
   3. $|f'(0)| \le \frac{3}{4}$ and $\left|f'\left(\frac{1}{2}\right)\right| \le \frac{4}{3}$
   4. $f(z)=z, z \in D$

   (CSIR NET June 2011)

2. Let f be a holomorphic function on $D=\{z \in \mathbb{C}:|z|<1\}$ such that $|f(z)| \le 1$. Define $g: D \to \mathbb{C}$ by
$$g(z)=\begin{cases} \frac{f(z)}{z} & \text{if } z \in D, z \neq 0, \\ f'(0) & \text{if } z=0 \end{cases}$$
   Which of the following statements are true ?
   1. g is holomorphic on D
   2. $|g(z)| \le 1$ for all $z \in D$
   3. $|f'(z)| \le 1$ for all $z \in D$
   4. $|f'(0)| \le 1$ **(CSIR NET Dec 2011)**

3. Let $f: D \to D$ be holomorphic with $f(0)=0$ and $f\left(\frac{1}{2}\right)=0$, where $D=\{z:|z|<1\}$. Which of the following statements are correct ?
   1. $\left|f'\left(\frac{1}{2}\right)\right| \le \frac{4}{3}$
   2. $|f'(0)| \le 1$
   3. $\left|f'\left(\frac{1}{2}\right)\right| \le \frac{4}{3}$ and $|f'(0)| \le 1$
   4. $f(z)=z, z \in D$

   (CSIR NET June 2012)

4. Let $D=\{z \in \mathbb{C}:|z|<1\}$. Which of the following are correct ?
   1. there exists a holomorphic function $f: D \to D$ with $f(0)=0$ and $f'(0)=2$
   2. there exists a holomorphic function $f: D \to D$ with $f\left(\frac{3}{4}\right)=\frac{3}{4}$ and $f'\left(\frac{2}{3}\right)=\frac{3}{4}$
   3. there exists a holomorphic function $f: D \to D$ with $f\left(\frac{3}{4}\right)=-\frac{3}{4}$ and $f'\left(\frac{3}{4}\right)=-\frac{3}{4}$
   4. there exists a holomorphic function $f: D \to D$ with $f\left(\frac{1}{2}\right)=-\frac{1}{2}$ and $f'\left(\frac{1}{4}\right)=1$

   (CSIR NET June 2013)

5. Let $D=\{z \in \mathbb{C}:|z|<1\}$. Then there exists a holomorphic function $f: D \to \bar{D}$ with $f(0)=0$ with the property
   1. $f'(0)=\frac{1}{2}$
   2. $\left|f\left(\frac{1}{3}\right)\right|=\frac{1}{4}$
   3. $f\left(\frac{1}{3}\right)=\frac{1}{2}$
   4. $f'(0)=\sec\left(\frac{\pi}{6}\right)$

   (CSIR NET June 2014)

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6. Let B be an open subset of $\mathbb{C}$ and ∂B denotes the boundary of B ; then which of following statements are correct :

1. for every entire function f we have $\partial(f(B)) \subseteq f(\partial B)$
2. for every entire function f and Q bounded open set B we have $\partial(f(B)) \subseteq f(\partial B)$
3. for every entire function f we have $\partial(f(B)) = f(\partial B)$
4. There exist an unbounded open subset B of $\mathbb{C}$ and an entire function f such that $\partial(f(B)) \subseteq f(\partial B)$

(CSIR NET Dec 2014)

7. Let f be an entire function. If Re f is bounded then
   1. Im f is constant
   2. f is constant
   3. $f \equiv 0$
   4. f' is a non zero constant

(CSIR NET June 2011)

8. Let f be an entire function. If Im $f \geq 0$, then
   1. Re f is constant
   2. f is constant
   3. $f \equiv 0$
   4. f' is a non zero constant

(CSIR NET June 2012)

9. Let f be an entire function. Which of the following statements are correct ?
   1. f is constant if the range of f is contained in a straight line.
   2. f is constant if f has uncountably many zero.
   3. f is constant if f is bounded on $\{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0\}$
   4. f is constant if the real part of f is bounded.

(CSIR NET June 2015)

10. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that $f = u + iv$ where u, v are the real and imaginary parts of f respectively. Then f is constant if
    1. $\{u(x,y) : z = x + iy \in \mathbb{C}\}$ is bounded.
    2. $\{v(x,y) : z = x + iy \in \mathbb{C}\}$ is bounded.
    3. $\{u(x,y) + v(x,y) : z = x + iy \in \mathbb{C}\}$ is bounded.
    4. $\{u^2(x,y) + v^2(x,y) : z = x + iy \in \mathbb{C}\}$ is bounded.

(CSIR NET June 2016)

11. Consider the polynomial $P(z) = \left(\sum_{n=0}^{5} a_n z^n\right)\left(\sum_{n=0}^{9} b_n z^n\right)$ where $a_n, b_n \in \mathbb{R}\ \forall n, a_5 \neq 0, b_9 \neq 0$. Then counting roots with multiplicity we can conclude that $P(z)$ has
    1. at least two real roots.
    2. 14 complex roots.
    3. no real roots.
    4. 12 complex roots.

(CSIR NET Dec 2016)

12. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. Let $g : \mathbb{D} \to \mathbb{D}$ be holomorphic, $g(0) = 0$, and let $h(z) = \begin{cases} g(z)/z, & z \in \mathbb{D}, z \neq 0 \\ g'(0), & z = 0 \end{cases}$.
    Which of the following statements are true?
    1. h is holomorphic in $\mathbb{D}$
    2. $h(\mathbb{D}) \subseteq \overline{\mathbb{D}}$.
    3. $|g'(0)| > 1$
    4. $|g(1/2)| \leq 1/2$

(CSIR NET Dec 2016)

13. Let f be an meromorphic function on $\mathbb{C}$ such that $|f(z)| \geq |z|$ at each point z when f is holomorphic then which of following is/are true ?
    1. The hypothesis are contrary so no such f exist
    2. there is a unique f satisfying given condition
    3. such f is an entire function
    4. there is a $A \in \mathbb{C}$ such that $|A| \geq 1$ and $f(z) = Az \quad \forall \quad z \in \mathbb{C}$

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14. Let $\mathbb{R}$ be the of all entire functions; then under pointwise addition and multiplication, which are true :
    1. the units in $\mathbb{R}$ are precisely the no-where vanishing entire function
    2. $\mathbb{R}$ is integral domain
    3. $\mathbb{R}$ is field
    4. All of these
15. Let $\mathbb{D}$ be the open unit disk centered at 0 in $\mathbb{C}$ and $f:\mathbb{D}\to\mathbb{C}$ be an analytic function. Let $f=u+iv$, where u,v are the real and imaginary parts of *f*. If $f(z)=\sum a_n z^n$ is the power series of *f*, then f is constant if
    1. f is analytic
    2. $u(1/2)\geq u(z)\ \ \forall\ z\in\mathbb{D}$
    3. The set $\{n\in\mathbb{N}\mid a_n=0\}$ is infinite
    4. For any closed curve γ in $\mathbb{D}$,

$$\int_{\gamma}\frac{f(z)dz}{(z-a)^2}=0\ \ \forall\ a\in\mathbb{D}\ \text{ with }\ |a|\geq 1/2$$

(CSIR NET June 2018)

True false key

1. F	2. T	3. T	4. T
5. F	6. T	7. T	8. T
9. T	10. T	11. T	12. T
13. T	14. T	15. F	16. T

Assignment key

S C Q

1. 4	2. 3	3. 3	4. 3
5. 2	6. 3	7. 3	8. 4
9. 2	10. 4	11. 3	12. 4
13. 2	14. 4	15. 3	16. 4
17. 2	18. 3	19. 4	20. 2
21. 2	22. 4	23. 2	24. 1
25. 2	26. 1	27. 2	28. 2
29. 3	30. 3	31. 4	32. 2
33. 2	34. 2	35. 3	

M C Q

1. 1,2,3	2. 1,2,4	3. 1,2,3
4. 2,3,4	5. 1,2	6. 2,4
7. 1,2	8. 1,2	9. 1,2,4
10. 1,2,3,4	11. 1,2 or 1,2,4	12. 1,2,4
13. 3,4	14. 1,2	15. 1,2,4

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